

LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING
LECTURE 1 - EXERCISES

Exercise 1. The goal is to give a polynomial-time reduction from the vertex-coloring problem to the stable set problem.

Let $G = (V, E)$ be a graph and let k be an integer. We build a new graph H as the *Cartesian product* of G and the clique K_k ; namely H is defined as follows.

Say, $V = \{v_1, \dots, v_n\}$. Let V_1, \dots, V_k be k disjoint copies of V , and set $V_h = \{v_{h,1}, \dots, v_{h,n}\}$ for $h = 1, \dots, k$. Then the vertex set of H is $V_1 \cup \dots \cup V_k$ and its edges are defined as follows: Two vertices $v_{h,i}$ and $v_{h,j}$ of V_h are adjacent in H if and only if v_i and v_j are adjacent in G (that is, we put a copy of G on each V_h); moreover, each set $\{v_{1,i}, v_{2,i}, \dots, v_{k,i}\}$ (consisting of the k copies of v_i) is a clique for each $i = 1, \dots, n$.

Show: $\chi(G) \leq k$ if and only if $\alpha(H) \geq n$.

Exercise 2. Let $G = (V, E)$ be a connected planar graph with $n \geq 3$ nodes. The goal is to show that $\chi(G) \leq 5$.

We let m denote the number of edges of G and f denote the number of faces in a plane drawing of G .

1) Show: $2m \geq 3f$.

2) Show: $m \leq 3n - 6$.

Hint: Use Euler formula: $n + f = m + 2$.

(For instance, for K_4 , $n = 4$, $m = 6$, and $f = 4$).

3) Show: G has at least one node u with degree at most 5.

3) Show: $\chi(G) \leq 5$ (using induction on n).

Hint: Distinguish two cases: (i) u has degree at most 4; (ii) u has degree 5.

In case (ii), pick two neighbors v_1 and v_2 of u that are not adjacent (**why do they exist?**) and consider the graph H obtained by contracting both edges $\{u, v_1\}$ and $\{u, v_2\}$, which is 5-colorable using induction.

Exercise 3. (Exercise 7.5 in the Lecture Notes). Derive König's edge cover theorem (Corollary 3.3a) from the Strong Perfect Graph Theorem.

The Strong Perfect Graph Theorem claims that if a graph H does not contain an odd cycle of length at least 5 or its complement as an induced subgraph then $\chi(H) = \omega(H)$.

König's edge cover theorem claims that, if G is a bipartite graph with no isolated vertex, then $\alpha(G) = \rho(G)$, where $\rho(G)$ is the minimum number of edges needed to cover all vertices of G .

Exercise 4. Let $G = (V, E)$ be a connected graph such that no minor of G is isomorphic to K_4 . The goal is to show that $\chi(G) \leq 3$.

We recall that a graph H is said to be a *minor* of G if H can be obtained from G by applying a sequence of the following operations: (i) deleting a vertex (and its adjacent edges), (ii) deleting an edge, (iii) contracting an edge. (For instance, if G contains a cycle of length at least 3 then K_3 is a minor of G .)

Call a set $S \subseteq V$ a *cutset* of G if G is connected but removing S from G disconnects the graph G . G is said to be *2-connected* if G is connected and it does not have a cutset S of cardinality $|S| = 1$.

1) Show: if G is connected with no K_4 -minor and G has at least 4 nodes, then G has a cutset S of cardinality at most 2.

Hint: Observe first that the result holds if G is a tree. So you may now assume that there exists a circuit C contained in G . Deal first with the case (i) when C contains all nodes of G and (ii) when C does not contain all nodes of G . In case (ii) use the fact that if a graph has no cutset of size at most 2 (i.e., G is 3-connected) then, by Menger's theorem, there are three internally vertex-disjoint paths between any two distinct vertices.

2) Assume G is 2-connected. Assume moreover that S is a cutset of G of cardinality 2, say, $S = \{a, b\}$, and assume that $\{a, b\}$ is not an edge of G . Let W_1, \dots, W_k be the connected components of $G \setminus S$. For each $i \in [k]$, consider the graph H_i obtained by adding the edge $\{a, b\}$ to $G[W_i \cup S]$ (the subgraph of G induced by $S \cup W_i$). Show that H_i does not contain K_4 as a minor.

3) Show: if G is connected with no K_4 -minor, then $\chi(G) \leq 3$.