LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING
LECTURE 2 (14/09/20) - EXERCISES

Exercise 1. Let $G = (V, E)$ be a connected planar graph with $n \geq 3$ nodes. The goal is to show that $\chi(G) \leq 5$.

We let m denote the number of edges of G and f denote the number of faces in a plane drawing of G.

1) Show: $2m \geq 3f$.

2) Show: $m \leq 3n - 6$.

Hint: Use Euler formula: $n + f = m + 2$.
(For instance, for K_4, $n = 4$, $m = 6$, and $f = 4$).

3) Show: G has at least one node u with degree at most 5.

3) Show: $\chi(G) \leq 5$ (using induction on n).

Hint: Distinguish two cases: (i) u has degree at most 4; (ii) u has degree 5.
*In case (ii), pick two neighbors v_1 and v_2 of u that are not adjacent (**why do they exist?**) and consider the graph H obtained by contracting both edges $\{u, v_1\}$ and $\{u, v_2\}$, which is 5-colorable using induction.*

Exercise 2. (Exercise 7.4 in the Lecture Notes). Derive from König's edge cover theorem (Corollary 3.3a) that if G is the complement of a bipartite graph, then $\omega(G) = \chi(G)$.

Note: The symbol $\gamma(G)$ is used in the Lecture Notes for denoting the coloring number of G, which is denoted here as $\chi(G)$.

König's edge cover theorem claims that, if G is a bipartite graph with no isolated vertex, then $\alpha(G) = \rho(G)$, where $\rho(G)$ is the minimum number of edges needed to cover all vertices of G.

Exercise 3. (Exercise 7.5 in the Lecture Notes). Derive König's edge cover theorem (Corollary 3.3a) from the Strong Perfect Graph Theorem.

The Strong Perfect Graph Theorem claims that if a graph H does not contain an odd cycle of length at least 5 or its complement as an induced subgraph then $\chi(H) = \omega(H)$.

Exercise 4. (Exercise 7.6 in the Lecture Notes). Let H be a bipartite graph and let G be the complement of the line-graph of H. Derive from König's matching theorem (Theorem 3.3) that $\omega(G) = \chi(G)$.

The *line graph* of a graph $H = (V, E)$ is the graph L_H with vertex set E and with edges the pairs $\{e, e'\} \subseteq E$ such that $|e \cap e'| = 1$.