Exercises – 25 November 2022
LNMB Course Networks and Semidefinite Programming

Note: In what follows the symbol $\chi(\cdot)$ is used to denote the vertex-coloring number of a graph, which is denoted by $\gamma(\cdot)$ in [LNAS].

Exercise 1. Let $G = (V, E)$ be a connected simple planar graph with $n \geq 3$ nodes. The goal is to show that $\chi(G) \leq 5$.

We let m denote the number of edges of G and f denote the number of faces in a plane drawing of G.

(1) Show: $2m \geq 3f$.

(2) Show: $m \leq 3n - 6$.

Hint: Use Euler's formula: $n + f = m + 2$.
(For instance, for K_4, $n = 4$, $m = 6$ and $f = 4$).

(3) Show: G has at least one node u with degree at most 5.

(4) Show: $\chi(G) \leq 5$ (using induction on n).

Hint: Distinguish two cases: (i) u has degree at most 4; (ii) u has degree 5. In case (ii), pick two neighbors v_1 and v_2 of u that are not adjacent (why do they exist?) and consider the graph H obtained by contracting both edges $\{u, v_1\}$ and $\{u, v_2\}$, which is 5-colorable using induction.

Exercise 2. (Exercise 7.5 in [LNAS]). Derive König's edge cover theorem (Corollary 3.3a in [LNAS]) from the Strong Perfect Graph Theorem.

Note: The Strong Perfect Graph Theorem claims that if a graph H does not contain an odd cycle of length at least 5 or its complement as an induced subgraph, then its vertex-colouring number $\chi(H)$ is equal to its clique number $\omega(H)$.

Exercise 3. (Exercise 7.12 in [LNAS]). Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be partitions of a finite set X such that $|A_1| = \ldots = |A_n| = |B_1| = \ldots = |B_n| = k$. Show that $\mathcal{A}$ and $\mathcal{B}$ have k disjoint common transversals.

Note: A set $Y \subseteq X$ is called a *transversal* of $\mathcal{A}$ if $|Y \cap A_i| = 1$ for all $i \in [n]$.
Hint: Note that König's edge-coloring theorem (Theorem 7.3 in [LNAS]) holds for bipartite graphs with multiple edges.