Exercises – 27 January 2025
LNMB Course Networks and Semidefinite Programming

Exercise A. Let $G = (V = [n], E)$ be a graph with nonnegative edge weights $w \in \mathbb{R}^E_+$, and extend w to the edge set of the complete graph K_n by setting $w_{ij} = 0$ if $i \neq j \in V$ and $\{i, j\} \notin E$. Define the associated Laplacian matrix $L \in \mathcal{S}^n$, with entries

$$L_{ii} = \sum_{j \in V \setminus \{i\}} w_{ij} \quad \text{for } i \in V,$$
$$L_{ij} = L_{ji} = -w_{ij} \quad \text{for } i \neq j \in V.$$

(a) Show that $x^T L x = \sum_{1 \le i < j \le n} w_{ij}(x_i - x_j)^2$ for any $x \in \mathbb{R}^n$, and that $L \succeq 0$.

(b) Show that $\frac{1}{4} x^T L x = \sum_{1 \le i < j \le n} w_{ij} \big(\frac{1 - x_i x_j}{2} \big)$ for any $x \in \{\pm 1\}^n$.

Note: The above underlies the alternative formulation (4.8) for the max-cut problem and the alternative formulation (4.9) for its sdp relaxation. After taking the dual one arrives at the dual formulation (4.11), which reads

$$\begin{aligned} \text{sdp}(G, w) &= \min \Big\{ \frac{n}{4} \lambda_{\max}(L + \text{Diag}(u)) : u \in \mathbb{R}^n, \sum_{i=1}^n u_i = 0 \Big\} \\ &= \max \Big\{ \frac{n}{4} t : tI - L - \text{Diag}(u) \succeq 0, \ t \in \mathbb{R}, \ u \in \mathbb{R}^n, \sum_{i=1}^n u_i = 0 \Big\}. \end{aligned} \tag{1}$$

Here, $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue. For $u \in \mathbb{R}^n$, $\text{Diag}(u)$ is the diagonal matrix whose diagonal entries are $u_1, \ldots, u_n$. (Please read the proof of Theorem 4.2.2.)

Exercise B. Consider the complete graph K_n, where all edges have weight 1.

(a) Compute the value of $\text{mc}(K_n)$, the maximum cardinality of a cut in K_n.

(b) Compute the optimal value of the semidefinite program

$$\min\{\langle J, X \rangle : X \in \mathcal{S}^n, \ X \succeq 0, \ X_{ii} = 1 \text{ for } i \in [n]\}.$$

(c) Compute the value of $\text{sdp}(K_n)$, the optimal value of the semidefinite relaxation for K_n equipped with all-ones edge weights.

Exercise C. Let $G = (V = [n], E)$ be a graph, equipped with all-ones edge weights (i.e., $w_{ij} = 1$ for $\{i, j\} \in E$). As in Exercise A, consider the associated Laplacian matrix L and the formulation for $\text{sdp}(G)$ from relation (1).

(a) Assume G is a vertex-transitive graph. Then, G is r-regular for some integer r.

- (a1) Show that $L = rI_n - A_G$, where A_G is the adjacency matrix of G (with entries $(A_G)_{ij} = 1$ if $\{i, j\} \in E$, and entries 0 elsewhere).
- (a2) Show: If σ is a permutation of V that is an automorphism of G, then $\sigma(A_G) = A_G$. Recall that $\sigma(A_G) = ((A_G)_{\sigma(i)\sigma(j)})_{i,j=1}^n$.

(a3) Show that $\mathrm{sdp}(G) = \frac{n}{4}\lambda_{\max}(L)$.
Hint: Use the formulation (1) and show that $u = 0$ is an optimal solution.

(b) Assume $G = C_{2n+1}$ is an odd cycle, with $n \geq 1$. Show that

$$\mathrm{sdp}(G) = \frac{2n+1}{2}\Big(1 + \cos\big(\frac{\pi}{2n+1}\big)\Big).$$

Hint: The eigenvalues of the adjacency matrix of C_{2n+1} are $2\cos\big(\frac{2k\pi}{2n+1}\big)$ for $0 \leq k \leq 2n$.