Exercises – 6 January 2025
LNMB Course Networks and Semidefinite Programming

Exercise A. Let $C \in \mathcal{S}^n$ be a real symmetric matrix with spectral decomposition $C = \sum_{i=1}^n \lambda_i u_i u_i^T$, where the u_i's form an orthonormal basis with u_i an eigenvector of C corresponding to eigenvalue λ_i. Assume $\lambda_1 \geq \ldots \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \ldots \geq \lambda_n$ and let $\sigma_+(C) := \lambda_1 + \ldots + \lambda_p$ denote the sum of the positive eigenvalues of C. Consider the following semidefinite programs

$$\text{sdp}_1(C) := \max\{\langle C, X\rangle : X \in \mathcal{S}^n,\ X \succeq 0,\ I_n - X \succeq 0\}, \tag{1}$$

$$\text{sdp}_2(C) := \min\{\text{Tr}(Y) : Y \in \mathcal{S}^n,\ Y \succeq 0,\ Y - C \succeq 0\}, \tag{2}$$

where I_n is the identity matrix.

(a) Show that $\sigma_+(C) \leq \text{sdp}_1(C)$.

(b) Show that $\text{sdp}_1(C) \leq \text{sdp}_2(C)$.

(c) Show that $\text{sdp}_2(C) \leq \sigma_+(C)$.

Exercise B. The goal of the exercise is to give an alternative proof to Lemma 3.4.4 (different from the proof in the Lecture Notes [NSP]). Let $G = (V = [n], E)$ be a graph. Recall the definition of the theta parameter $\vartheta(G)$ from relation (3.13):

$$\vartheta(G) = \max\{\langle J, X\rangle : X \in \mathcal{S}^n,\ X \succeq 0,\ \text{Tr}(X) = 1,\ X_{ij} = 0 \text{ if } \{i,j\} \in E\}. \tag{3}$$

Define the parameter

$$\vartheta_1(G) := \max\left\{\text{Tr}(X) : X \in \mathcal{S}^n, x \in \mathbb{R}^n, \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0,\ x_i = X_{ii} \text{ for } i \in [n],\ X_{ij} = 0 \text{ if } \{i,j\} \in E\right\} \tag{4}$$

(which coincides with the parameter defined in relation (3.19)).

(a) Show that $\vartheta_1(G) \leq \vartheta(G)$.

(b) Show that $\vartheta(G) \leq \vartheta_1(G)$.

Steps for showing (b): Let X be an optimal solution for the program (3), and let $u_1, \ldots, u_n \in \mathbb{R}^n$ be vectors such that $X_{ij} = u_i^T u_j$ for $i, j \in [n]$ (why do they exist?). Define the vectors

$$d = \sum_{i=1}^n u_i, \quad v_0 = \frac{d}{\|d\|}, \quad v_i = \frac{d^T u_i}{\|d\| \cdot \|u_i\|^2} u_i \ \text{ for } i \in [n].$$

(b1) Show that $\sum_{i=1}^n \|u_i\|^2 = 1$ and $\vartheta(G) = \|d\|^2$.

(b2) Show that the matrix $Y = (v_i^T v_j)_{i,j=0,1,\ldots,n}$ provides a feasible solution for the program (4), i.e., that it is of the form $Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ where x, X satisfy the constraints of (4).

(b3) Show that $\|d\|^2 \leq \sum_{i=1}^n \|v_i\|^2 \leq \vartheta_1(G)$.

Hint: Show that $\|d\|^4 = (\sum_{i=1}^n d^T u_i)^2 \leq \sum_{i=1}^n \left(\frac{d^T u_i}{\|u_i\|}\right)^2$ (using Cauchy-Schwartz inequality).