Exercises A & B – 18 November 2024
LNMB Course Networks and Semidefinite Programming

Exercise A. The goal is to give a polynomial-time reduction from the vertex-coloring problem to the stable set problem.

Consider an instance of the coloring problem, consisting of a graph $G = (V, E)$ and an integer $k \geq 1$, and the task is to decide whether there exists a vertex-coloring of G using k colours, i.e., whether $\chi(G) \leq k$.

We build a new graph H as the Cartesian product of G and the complete graph K_k on k nodes. That is, the vertices of H are the pairs (u, i), where $u \in V$ and $i \in [k]$, and two distinct vertices (u, i) and (v, j) are adjacent in H if and only if either $\{u, v\} \in E$ and $i = j$, or $u = v$ and $i \neq j$ (i.e., $\{i, j\}$ is an edge in K_k).

Show that

$$\chi(G) \leq k \iff \alpha(H) \geq n.$$

Exercise B. The goal is to give a polynomial-time reduction from the stable set problem to the vertex-coloring problem.

Consider an instance of the stable set problem, consisting of a graph $G = (V, E)$ and an integer $k \geq 1$, and the task is to decide whether there exists a stable set of G of cardinality k, i.e., whether $\alpha(G) \geq k$.

We build a new graph $H = (W, F)$ in the following manner.

Let V' be a disjoint copy of the vertex set V of G, and let U be a set of cardinality k that is disjoint from V and V'. Set $V = \{v_1, \ldots, v_n\}$, $V' = \{v'_1, \ldots, v'_n\}$, and $U = \{u_1, \ldots, u_k\}$. Then, the vertex set of H is $W = V \cup V' \cup U$, with $|W| = 2n + k$.

The edge set of H consists of the following edges:

- Two distinct nodes of V are adjacent in H if and only if they are adjacent in G (i.e., there is a copy of G on V).
- Any two distinct nodes of V' are adjacent (i.e., there is a complete graph on V').
- Any two distinct nodes of U are adjacent (i.e., there is a complete graph on U).
- Two nodes $v_i \in V$ and $v'_j \in V'$ are adjacent if and only if $i \neq j$ (i.e., each node $v_i \in V$ is adjacent to each node $v'_j \in V'$ except to its 'copy' v'_i).
- Every node of V is adjacent to every node of U (i.e., there is a complete bipartite graph between V and U).

Show that

$$\alpha(G) \geq k \iff \chi(H) \leq n + 1.$$