Geometric packing and coloring problems

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joint work with

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References

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  Lower bounds for measurable chromatic numbers,
  (Preprint on arXiv:0801.1059v2 [math.CO], 18 pages)
  to appear in Geometric and Functional Analysis

- F.M. de Oliveira Filho, F. Vallentin,
  Fourier analysis, linear programming and densities of distance avoiding sets in $\mathbb{R}^n$,
  (Preprint on arXiv:0808.1822v1 [math.CO], 10 pages)
  to appear in Journal of the European Mathematical Society

Motivation

In the last years I studied packing problems in compact metric spaces, e.g. the kissing number problem.

Tools: combinatorial optimization in a continuous setting:
- semidefinite optimization
- harmonic analysis

Questions:
- Dealing with non-compact spaces?
- From discrete point sets to sets with positive measure?

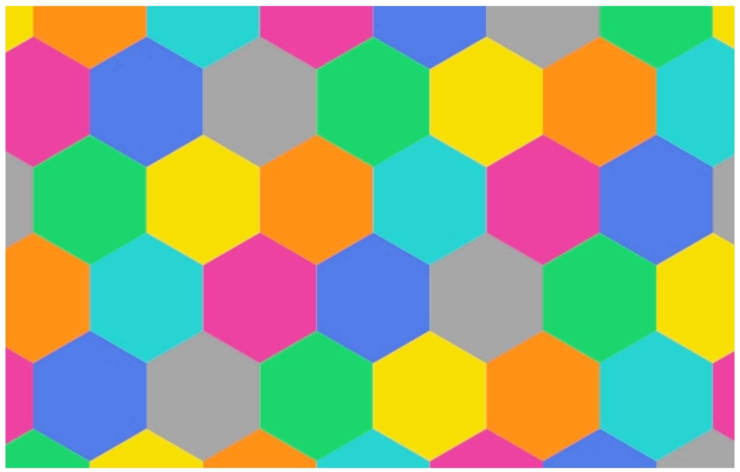
Chromatic number of Euclidean space

$\chi(\mathbb{R}^n) =$ minimal number of colors needed to paint all points in $\mathbb{R}^n$ so that every two points at distance 1 receive different colors.

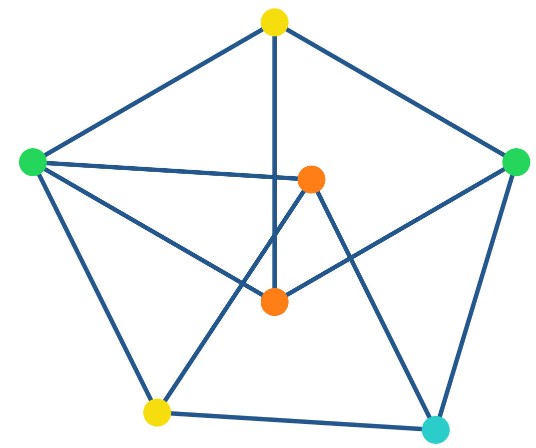
$\chi_m(\mathbb{R}^n) =$ minimal number colors needed to paint all points in $\mathbb{R}^n$ so that every two points at distance 1 receive different colors. & points receiving the same colors are Lebesgue measurable sets.

Clearly, $\chi(\mathbb{R}^n) \leq \chi_m(\mathbb{R}^n)$, and maybe $\chi(\mathbb{R}^n) < \chi_m(\mathbb{R}^n)$ for some n.

What is known?



$7 \geq \chi_m(\mathbb{R}^2)$



$\chi(\mathbb{R}^2) \geq 4$

1000$ problem

(Graham/Erdős)

Falconer (1981): $\chi_m(\mathbb{R}^2) \geq 5$

Frankl, Wilson (1981), Larman, Rogers (1972):

$$(1 - o(1))1.2^n \leq \chi(\mathbb{R}^n) \leq \chi_m(\mathbb{R}^n) \leq (3 + o(1))^n$$

Distance avoiding sets

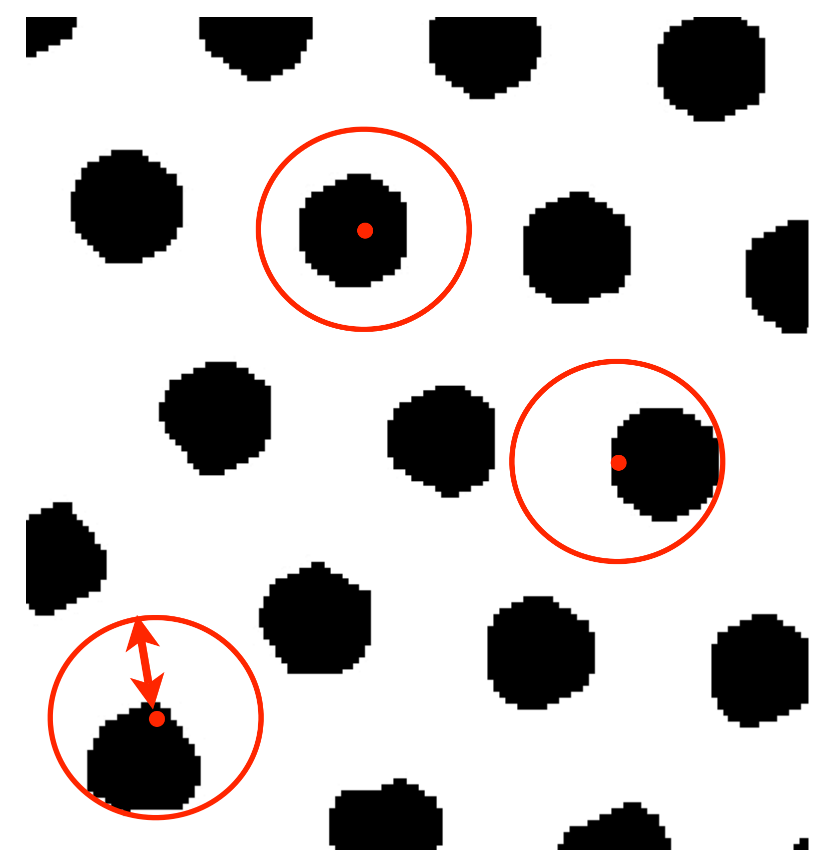
$A \subseteq \mathbb{R}^n$ measurable set

A avoids the distances $d_1, \ldots, d_N$ if

$$\forall x, y \in A : \|x - y\| \notin \{d_1, \ldots, d_N\}.$$

A has upper density

$$\overline{\delta}(A) = \limsup_{T \to \infty} \frac{\operatorname{vol}(A \cap [-T, T]^n)}{\operatorname{vol}[-T, T]^n}.$$



$\overline{\delta}(A) \approx 0.200544$

Relation

- extreme density

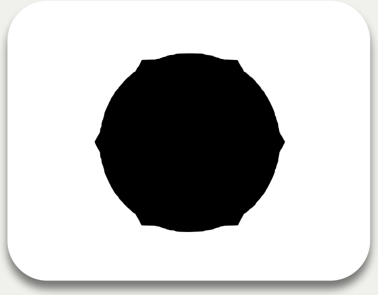
$$m_{d_1,\ldots,d_N}(\mathbb{R}^n) = \sup\{\overline{\delta}(A) : A \subseteq \mathbb{R}^n \text{ avoids } d_1, \ldots, d_N\}$$

- Upper bounds for $m_1(\mathbb{R}^n)$ give lower bounds for $\chi_m(\mathbb{R}^n)$:

$$m_1(\mathbb{R}^n) \cdot \chi_m(\mathbb{R}^n) \geq 1$$

- Conjecture (Erdős): $m_1(\mathbb{R}^2) < 0.25$

  Best bounds (previously) known (Croft (1967), Székely (1984)):



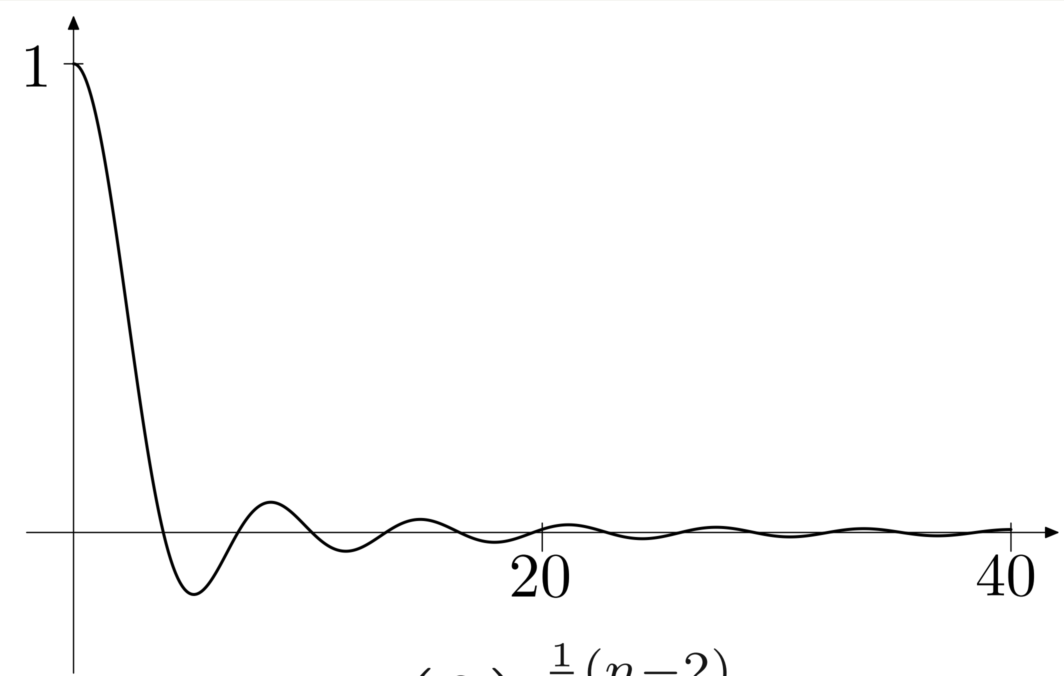
$$0.2293 \leq m_1(\mathbb{R}^2) \leq 0.2790$$

The linear programming bound

Main Theorem. Suppose that there are real numbers $z_0, z_1, \ldots, z_N$ which satisfy $z_0 + z_1 + \cdots + z_N = 1$ and that for all $t \geq 0$

$$z_0 + \Omega_n(d_1 t) z_1 + \Omega_n(d_2 t) z_2 + \cdots + \Omega_n(d_N t) z_N \geq 0$$

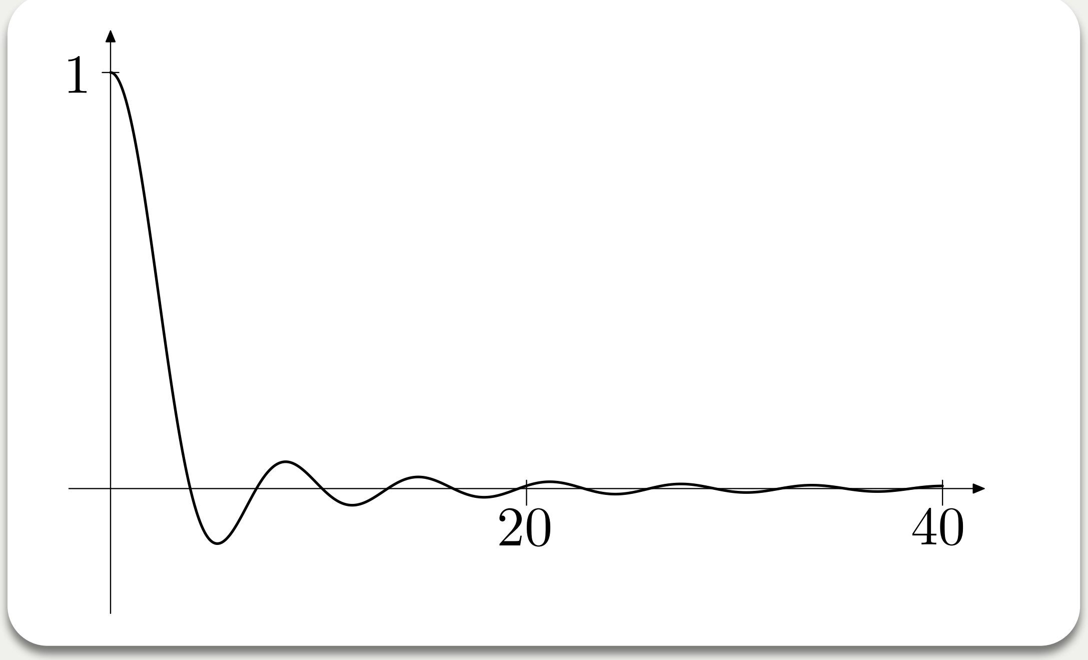
holds, then $m_{d_1,\ldots,d_N}(\mathbb{R}^n) \leq z_0$.



$$\Omega_n(x) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{x}\right)^{\frac{1}{2}(n-2)} J_{\frac{1}{2}(n-2)}(x)$$
$$= \int_{S^{n-1}} e^{iu\cdot\xi}\, d\omega(\xi), \quad x = \|u\|$$

Application I- one distance

- **LP**: $\min\{z_0 : z_0 + z_1 = 1, z_0 + \Omega_n(t) z_1 \geq 0, t \geq 0\} \geq m_1(\mathbb{R}^n)$

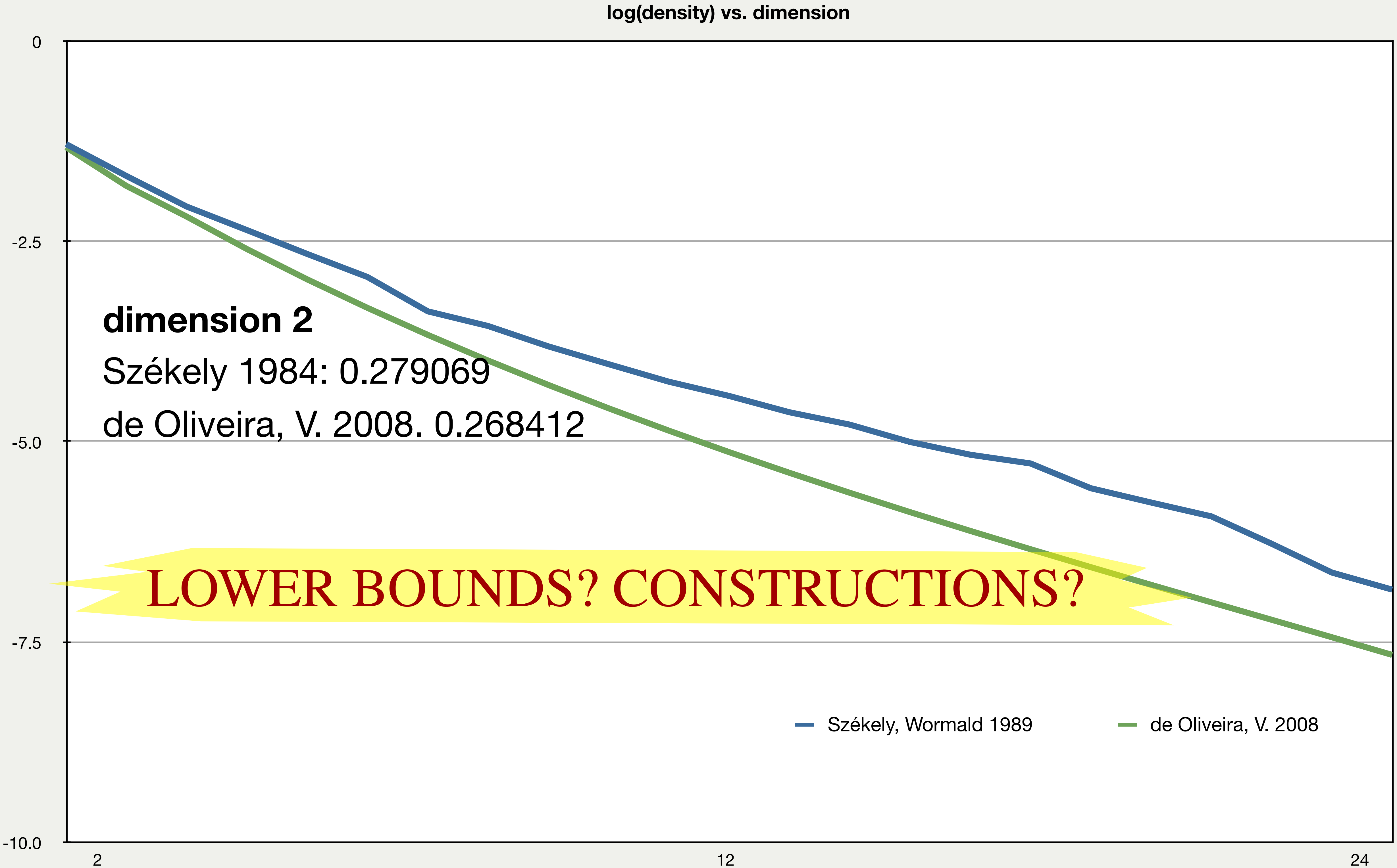


- **Solution**: $z_0 + z_1 = 1$ and $z_0 + \Omega_n(j_{n,1}) z_1 = 0$

$j_{n,1}$ — first minimum of Ω_n

- **Corollary**: $\chi_m(\mathbb{R}^n) \geq 1 - \dfrac{1}{\Omega_n(j_{n,1})} \sim 1.165^n$

Computational results



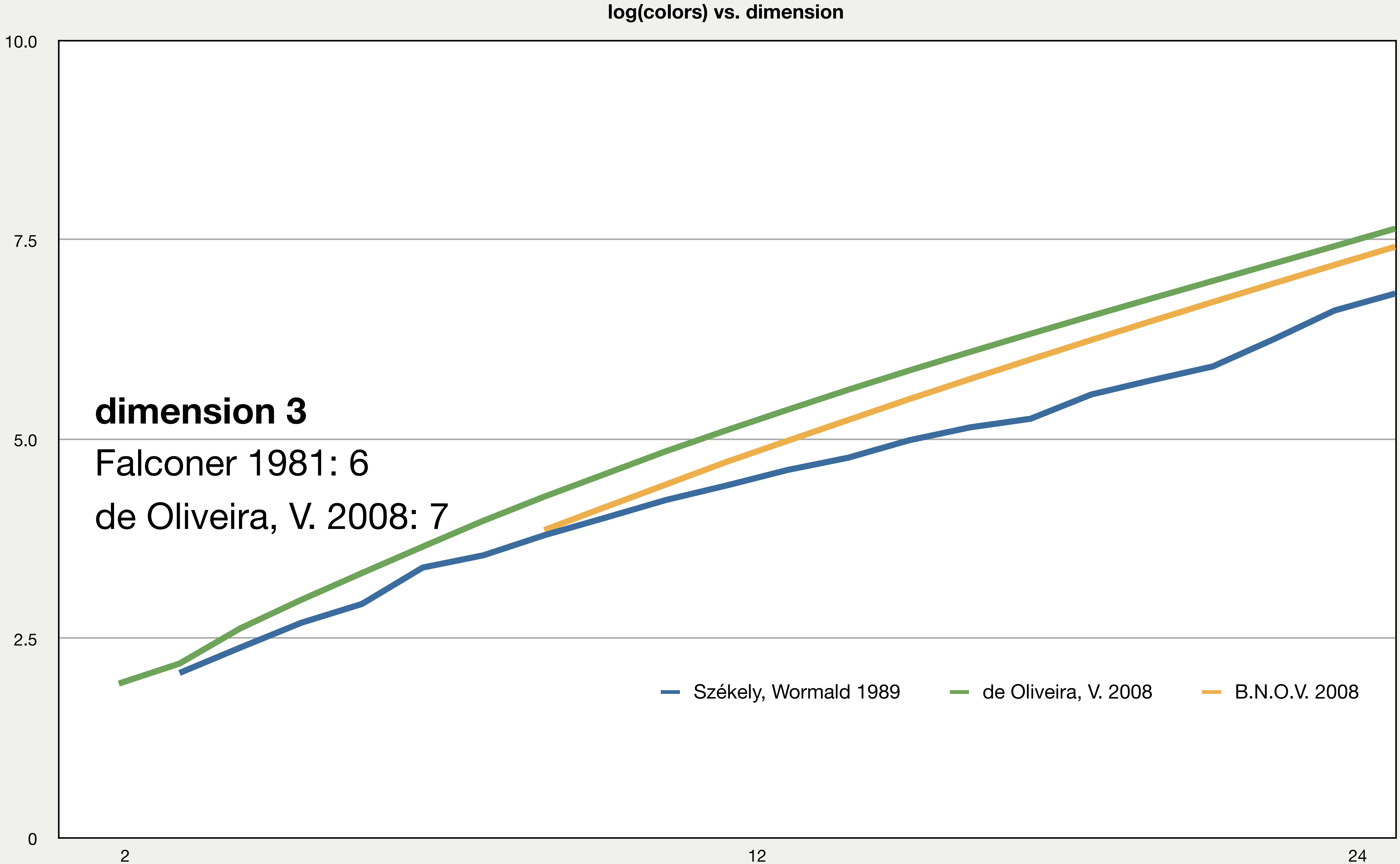
dimension 2

Székely 1984: 0.279069

de Oliveira, V. 2008. 0.268412

LOWER BOUNDS? CONSTRUCTIONS?

Computational results



Application 2 - many distances

- Can be used to prove a quantitative version of

Theorem. (Furstenberg, Katznelson, Weiss, 1990)
Let $A \subset \mathbb{R}^2$ be a measurable set with $\overline{\delta}(A) > 0$. Then, there is a d_0 so that for all $d \geq d_0$ the set A does not avoid the distance d.

- Their proof uses ergodic theory. Other proofs:
  Bourgain (1986) — harmonic analysis
  Falconer, Marstrand (1986) — geometric measure theory
  Bukh (2007) — Szemerédi's regularity lemma

Proof sketch of main theorem

- Simplifications: only avoid distance $d_1 = 1$
  (saves some notation)

  don't care about existence of limits
  (saves 10 minutes)

Fourier analysis

$f, g : \mathbb{R}^n \to \mathbb{C}$ complex-valued functions

$$(f, g) = \lim_{T\to\infty} \frac{1}{\mathrm{vol}[-T, T]^n} \int_{[-T,T]^n} f(x)\overline{g(x)}dx$$

mean value inner product

for $u \in \mathbb{R}^n$: $\widehat{f}(u) = (f, e^{iu\cdot x})$ ($\widehat{f}$ has discrete support in our case)

Fourier coefficient

$$(f, g) = \sum_{u\in\mathbb{R}^n} \widehat{f}(u)\overline{\widehat{g}(u)}$$

Parseval's formula

Autocorrelation function

$A \subset \mathbb{R}^n$ measurable set avoiding distance $d_1 = 1$

$$\varphi(x) = (1_A, 1_{A-x})$$

autocorrelation function of A

$$= \lim_{T\to\infty} \frac{1}{\text{vol}[-T,T]^n} \int_{[-T,T]^n} 1_A(y) 1_{A-x}(y)\,dy$$

$$= \delta(A \cap (A - x))$$

$$= \sum_{u \in \mathbb{R}^n} |\widehat{1_A}(u)|^2 e^{iu\cdot x}$$

$$\varphi(0) = \overline{\delta}(A)$$

$$\varphi(x) = 0 \text{ if } \|x\| = 1$$

$$\widehat{\varphi}(u) = |\widehat{1_A}(u)|^2 \geq 0$$

$$\widehat{\varphi}(0) = \overline{\delta}(A)^2$$

Linear programming

The linear program in the infinite variables $\widehat{\varphi}(u)$

$$\sup\left\{\frac{\widehat{\varphi}(0)}{\displaystyle\sum_{u\in\mathbb{R}^n}\widehat{\varphi}(u)} \quad : \quad \widehat{\varphi}(u)\geq 0,\ \ u\in\mathbb{R}^n,\right.$$
$$\left.\sum_{u\in\mathbb{R}^n}\widehat{\varphi}(u)e^{iu\cdot x}=0,\ \|x\|=1\right\}$$

gives an upper bound on $m_1(\mathbb{R}^n)$.

Symmetrization

Taking spherical averages simplifies its computation tremendously (and explains the appearance of Ω_n).

Transform φ into a radial function by taking averages over the unit sphere

For $x \in \mathbb{R}^n$ define

$$\int_{S^{n-1}} \varphi(\|x\|\xi)d\omega(\xi) = \int_{S^{n-1}} \sum_{u \in \mathbb{R}^n} \widehat{\varphi}(u) e^{iu \cdot \|x\| \xi} d\omega(\xi)$$

$$= \sum_{t \geq 0} \underbrace{\sum_{u : \|u\| = t} \widehat{\varphi}(u)}_{=: \alpha(t)} \underbrace{\int_{S^{n-1}} e^{iu \cdot \|x\| \xi} d\omega(\xi)}_{=: \Omega_n(\|x\| t)}$$

Duality

Symmetrization gives linear program in infinite variables $\alpha(t), t \in \mathbb{R}_{\geq 0}$.

$$\sup \left\{ \frac{\alpha(0)}{\sum_{t \geq 0} \alpha(t)} \quad : \quad \begin{array}{l} \alpha(t) \geq 0, \;\; t \geq 0, \\ \sum_{t \geq 0} \alpha(t) \Omega_n(t) = 0 \end{array} \right\}$$

Taking the dual linear program and showing weak duality proves the theorem.