



Fig. 0.1

Contents

1. Introduction	1
1.1 Symbols used in asymptotic estimates	1
1.2 Asymptotic expansions	2
1.3 A first example: Exponential integral	3
1.4 Generalized asymptotic expansions	5
1.5 Properties of asymptotic power series	6
1.6 Optimal truncation of asymptotic expansions	8
Basic Methods for Integrals	11
2. Expansions of Laplace-type integrals: Watson's lemma	13
2.1 Watson's lemma	13
2.1.1 Watson's lemma for extended sectors	14
2.1.2 More general forms of Watson's lemma	16
2.2 Watson's lemma for loop integrals	16
2.3 More general forms of Laplace-type integrals	19
2.3.1 Transformation to the standard form	19
2.4 How to compute the coefficients	20
2.4.1 Inversion method for computing the coefficients	20
2.4.2 Integrating by parts	22
2.4.3 Manipulating power series	23
2.4.4 Explicit forms of the coefficients in the expansion	25
2.5 Other kernels	26
2.6 Exponentially improved asymptotic expansions	27
2.7 Singularities of the integrand	29
2.7.1 A pole near the endpoint	29
2.7.2 More general cases	32
3. The method of Laplace	33

3.1	A theorem for the general case	33
3.2	Constructing the expansion	35
3.2.1	Inversion method for computing the coefficients	36
3.3	Explicit forms of the coefficients in the expansion	37
3.4	The complementary error function	38
4.	The saddle point method and paths of steepest descent	41
4.1	The axis of the valley at the saddle point	43
4.2	Examples with simple exponentials	43
4.2.1	A first example	43
4.2.2	A cosine transform	44
4.3	Steepest descent paths not through a saddle point	44
4.3.1	A gamma function example	45
4.3.2	An integral related to the error function	46
4.4	An example with strong oscillations: A 100–digits challenge	48
4.5	A Laplace inversion formula for $\operatorname{erfc} z$	49
4.6	A non-oscillatory integral for $\operatorname{erfc} z$, $z \in \mathbb{C}$	50
4.7	The complex Airy function	50
4.8	A parabolic cylinder function	53
5.	The Stokes phenomenon	57
5.1	The Airy function	57
5.2	The recent interest in the Stokes phenomenon	58
5.3	Exponentially small terms in the Airy expansions	59
5.4	Expansions in connection with the Stokes phenomenon	60
5.4.1	Applications to a Kummer function	61
	Basic Methods: Examples for Special Functions	63
6.	The gamma function	65
6.1	$\Gamma(z)$ by Laplace’s method	66
6.1.1	Calculating the coefficients	67
6.1.2	Details on the transformation	68
6.2	$1/\Gamma(z)$ by the saddle point method	71
6.2.1	Another integral representation of $1/\Gamma(z)$	72
6.3	The logarithm of the gamma function	72
6.3.1	Estimations of the remainder	73
6.4	Expansions of $\Gamma(z+a)$ and $1/\Gamma(z+a)$	75
6.5	The ratio of two gamma functions	76
6.5.1	A simple expansion	77
6.5.2	A more efficient expansion	78

Contents

v

6.6	A binomial coefficient	80
6.6.1	A uniform expansion of the binomial coefficient	83
6.7	Asymptotic expansion of a product of gamma functions	85
6.8	Expansions of ratios of three gamma functions	88
7.	Incomplete gamma functions	91
7.1	Integral representations	91
7.2	$\Gamma(a, x)$: asymptotic expansion for $x \gg a$	92
7.3	$\gamma(a, x)$: asymptotic expansion for $a > x$	93
7.4	$\Gamma(a, x)$: asymptotic expansion for $x > a$	97
8.	The Airy function	101
8.1	Expansions of $\text{Ai}(z), \text{Bi}(z)$	102
8.1.1	Transforming the saddle point contour	102
8.2	Expansions of $\text{Ai}(-z), \text{Bi}(-z)$	105
8.3	Two simple ways to obtain the coefficients	106
8.4	A generalized form of the Airy function	107
9.	Bessel functions: Large argument	109
9.1	The modified Bessel function $K_\nu(z)$	109
9.2	The ordinary Bessel functions	110
9.3	The modified Bessel function $I_\nu(z)$	111
9.3.1	A compound expansion of $I_\nu(z)$	111
9.4	Saddle point method for $K_\nu(z), z \in \mathbb{C}$	113
9.4.1	Integral representations from saddle point analysis	115
9.4.2	Saddle point method for $J_\nu(x), x \leq \nu$	116
9.5	Debye-type expansions of the modified Bessel functions	117
9.6	Modified Bessel functions of purely imaginary order	119
9.6.1	The monotonic case: $x \geq \nu \geq 0$	120
9.6.2	The oscillatory case: $\nu \geq x > 0$	123
9.7	A J -Bessel integral	126
10.	Kummer functions	129
10.1	General properties	129
10.2	Asymptotic expansions for large z	131
10.3	Expansions for large a	132
10.3.1	Tricomi's function $E_\nu(z)$	132
10.3.2	Expansion of $U(a, c, z), a \rightarrow +\infty$	133
10.3.3	Expansion of ${}_1F_1(a; c; z), a \rightarrow +\infty$	135
10.3.4	Expansion of ${}_1F_1(a; c; z), a \rightarrow -\infty$	137
10.3.5	Expansion of $U(a, c, z), a \rightarrow -\infty$	138
10.3.6	Slater's results for large a	139

10.4	Expansions for large c	142
10.4.1	Expansion of ${}_1F_1(a; c; z)$, $c \rightarrow +\infty$	142
10.4.2	Expansion of $U(a, c, z)$, $c \rightarrow +\infty$, $z < c$	142
10.4.3	Expansion of $U(a, c, z)$, $c \rightarrow +\infty$, $z > c$	144
10.4.4	Expansion of $U(a, c, z)$, $c \rightarrow -\infty$	144
10.4.5	Expansion of ${}_1F_1(a; c; z)$, $c \rightarrow -\infty$	146
10.5	Uniform expansions of the Kummer functions	146
11.	Parabolic cylinder functions: Large argument	149
11.1	A few properties of the parabolic cylinder functions	149
11.2	The function $U(a, z)$	150
11.3	The function $U(a, -z)$	152
11.4	The function $V(a, z)$	153
11.5	Expansions of the derivatives	154
12.	The Gauss hypergeometric function	155
12.1	Large values of c	156
12.1.1	Large positive c ; $ z \leq z_0$	156
12.1.2	Large negative c ; $ z \leq z_0$	157
12.1.3	Large positive c ; $ z \geq z_0$	157
12.1.4	Large negative c ; $ z \geq z_0$	158
12.2	Large values of b	158
12.2.1	Large negative b ; $ z \geq z_0$	158
12.2.2	Large negative b , $ z \leq z_0$	159
12.3	Other large parameter cases	159
12.3.1	Jacobi polynomials of large degree	160
12.3.2	An example of the case ${}_2F_1(a, b - \lambda; c + \lambda; z)$	162
13.	Examples of ${}_3F_2$ -polynomials	165
13.1	A ${}_3F_2$ associated with the Catalan-Larcombe-French sequence	165
13.1.1	Transformations	167
13.1.2	Asymptotic analysis	168
13.1.3	Asymptotic expansion	170
13.1.4	An alternative method	171
13.2	An integral of Laguerre polynomials	173
13.2.1	A first approach	173
13.2.2	A generating function approach	175
	Other Methods for Integrals	179
14.	The method of stationary phase	181

Contents

vii

14.1	Critical points	181
14.2	Integrating by parts: No stationary points	182
14.3	Three critical points: A formal approach	183
14.4	On the use of neutralizers	184
14.5	How to avoid neutralizers?	186
14.5.1	A few details about the Fresnel integral	188
14.6	Algebraic singularities at both endpoints: Erdélyi's example	189
14.6.1	Application: A conical function	190
14.6.2	Avoiding neutralizers in Erdélyi's example	191
14.7	Transformation to standard form	192
14.8	General order stationary points	194
14.8.1	Integrating by parts	194
14.9	The method fails: Examples	195
14.9.1	The Airy function	196
14.9.2	A more complicated example	196
15.	Coefficients of a power series. Darboux's method	201
15.1	A first example: A binomial coefficient	202
15.2	Legendre polynomials of large degree	203
15.2.1	A paradox in asymptotics	205
15.3	Gegenbauer polynomials of large degree	206
15.4	Jacobi polynomials of large degree	207
15.5	Laguerre polynomials of large degree	207
15.6	Generalized Bernoulli polynomials $B_n^{(\mu)}(z)$	208
15.6.1	Asymptotic expansions for large degree	209
15.6.2	An alternative expansion	211
15.7	Generalized Euler polynomials $E_n^{(\mu)}(z)$	213
15.7.1	Asymptotic expansions for large degree	213
15.7.2	An alternative expansion	214
15.8	Coefficients of expansions of the ${}_1F_1$ -function	216
15.8.1	Coefficients of Tricomi's expansion	216
15.8.2	Coefficients of Buchholz's expansion	219
16.	Mellin-Barnes integrals and Mellin convolution integrals	223
16.1	Mellin-Barnes integrals	224
16.2	Mellin convolution integrals	226
16.3	Error bounds	228
17.	Alternative expansions of Laplace-type integrals	229
17.1	Hadamard-type expansions	229
17.2	An expansion in terms of Kummer functions	231
17.3	An expansion in terms of factorial series	232

17.4	The Franklin–Friedman expansion	235
18.	Two-point Taylor expansions	239
18.1	The expansions	240
18.2	An alternative form of the expansion	241
18.3	Explicit forms of the coefficients	242
18.4	Manipulations with two-point Taylor expansions	243
19.	Hermite polynomials as limits of other classical orthogonal polynomials	247
19.1	Limits between orthogonal polynomials	247
19.2	The Askey scheme of orthogonal polynomials	249
19.3	Asymptotic representations	249
19.4	Gegenbauer polynomials	251
19.5	Laguerre polynomials	252
19.6	Generalized Bessel polynomials	253
19.7	Meixner–Pollaczek polynomials into Laguerre polynomials	255
	Uniform Methods for Integrals	257
20.	An overview of standard forms	259
20.1	Comments on the table	261
21.	A saddle point near a pole	265
21.1	A saddle point near a pole: Van der Waerden’s method	265
21.2	An alternative expansion	267
21.3	An example from De Bruijn	268
21.4	A pole near a double saddle point	269
21.5	A singular perturbation problem and K –Bessel integrals	270
21.5.1	A Bessel K_0 –integral	270
21.5.2	A similar Bessel K_1 –integral	272
21.5.3	A singular perturbation problem	273
21.6	A double integral with poles near saddle points	275
21.6.1	Application to a singular perturbation problem	276
21.7	The Fermi–Dirac integral	279
22.	Saddle point near algebraic singularity	283
22.1	A saddle point near an endpoint of the interval	283
22.2	The Bleistein expansion	284
22.3	Extending the role of the parameter β	287
22.4	Contour integrals	289
22.5	Kummer functions in terms of parabolic cylinder functions	290

Contents

ix

22.5.1	Uniform expansion of $U(a, c, z)$, $c \rightarrow +\infty$	291
22.5.2	Uniform expansion of ${}_1F_1(a; c; z)$, $c \rightarrow +\infty$	294
23.	Two coalescing saddle points: Airy-type expansions	297
23.1	The standard form	297
23.2	An integration by parts method	298
23.3	How to compute the coefficients	300
23.4	An Airy-type expansion of the Hermite polynomial	303
23.4.1	The cubic transformation	304
23.4.2	Details on the coefficients	306
23.5	An Airy-type expansion of the Bessel function $J_\nu(z)$	307
23.6	A semi-infinite interval: Incomplete Scorer function	311
23.6.1	A singular perturbation problem inside a circle	313
24.	Hermite-type expansions of integrals	317
24.1	An expansion in terms of Hermite polynomials	318
24.1.1	Cauchy-type integrals for the coefficients	319
24.2	Gegenbauer polynomials	321
24.2.1	Preliminary steps	322
24.2.2	A first approximation	323
24.2.3	Transformation to the standard form	324
24.2.4	Special cases of the expansion	329
24.2.5	Approximating the zeros	330
24.2.6	The relativistic Hermite polynomials	331
24.3	Tricomi–Carlitz polynomials	331
24.3.1	Contour integral and saddle points	333
24.3.2	A first approximation	335
24.3.3	Transformation to the standard form	335
24.3.4	Approximating the zeros	337
24.4	More examples	338

Uniform Methods for Laplace-Type Integrals 339

25.	The vanishing saddle point	341
25.1	Expanding at the saddle point	342
25.2	An integration by parts method	344
25.2.1	Representing coefficients as a Cauchy-type integral	345
25.3	Expansions for loop integrals	346
25.4	Kummer functions	348
25.5	Generalized zeta function	348
25.6	Transforming to the standard form	349

25.6.1	The ratio of two gamma functions	350
25.6.2	Parabolic cylinder functions	352
26.	A moving endpoint: Incomplete Laplace integrals	353
26.1	The standard form	353
26.2	Constructing the expansion	354
26.2.1	The complementary function	355
26.3	Application to the incomplete beta function	356
26.3.1	Expansions of the coefficients	359
26.4	A corresponding loop integral	360
26.4.1	Application to the incomplete beta function	361
27.	An essential singularity: Bessel-type expansions	363
27.1	Expansions in terms of modified Bessel functions	363
27.2	A corresponding loop integral	366
27.3	Expansion at the internal saddle point	366
27.4	Application to Kummer functions	367
27.4.1	Expansion of $U(a, c, z)$, $a \rightarrow +\infty$, $z > 0$	367
27.4.2	Auxiliary expansions and further details	370
27.4.3	Expansion of ${}_1F_1(a; c; z)$, $a \rightarrow +\infty$, $z \geq 0$	372
27.4.4	Expansion of ${}_1F_1(a; c; z)$, $a \rightarrow -\infty$, $0 \leq z < -4a$	373
27.4.5	Expansion of $U(a, c, z)$, $a \rightarrow -\infty$, $0 < z < -4a$,	375
27.5	A second uniformity parameter	377
27.5.1	Expansion of $U(a, c, z)$, $a \rightarrow \infty$, $z > 0$, $c \leq 1$	378
27.5.2	Expansion of ${}_1F_1(a; c; z)$, $a \rightarrow \infty$, $z \geq 0$, $c \geq 1$	380
28.	Expansions in terms of Kummer functions	381
28.1	Approximation in terms of the Kummer U -function	381
28.1.1	Constructing the expansions	382
28.1.2	Expansion for the loop integral	385
28.2	The ${}_2F_1$ -function, large c : Kummer U	385
28.2.1	Legendre polynomials: uniform expansions	386
28.3	The ${}_2F_1$ -function, large b : Kummer ${}_1F_1$	387
28.3.1	Using a real integral	388
28.3.2	Using a loop integral	392
28.4	Jacobi polynomials of large degree: Laguerre-type expansion	392
28.4.1	Laguerre-type expansion for large values of β	396
28.5	Expansion of a Dirichlet-type integral	398

Uniform Methods: Examples for Special Functions	401
29. Legendre functions	403
29.1 Expansions of $P_\nu^\mu(z), Q_\nu^\mu(z)$; $\nu \rightarrow \infty, z \geq 1$	404
29.1.1 Expansions for $z \geq z_0 > 1$	404
29.1.2 Expansion in terms of modified Bessel functions	405
29.1.3 Expansions of $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ in terms of Bessel functions	409
29.2 Expansions of $P_\nu^\mu(z), Q_\nu^\mu(z)$; $\mu \rightarrow \infty, z > 1$	410
29.2.1 Expansions for bounded z	410
29.2.2 Expansions in terms of modified Bessel functions	410
29.2.3 Expansions of $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$	411
29.3 Integrals with nearly coincident branch points	412
29.3.1 Ursell's expansions of Legendre functions	413
29.3.2 An alternative expansion of $P_n^{-m}(\cosh z)$	415
29.4 Toroidal harmonics and conical functions	416
30. Parabolic cylinder functions: Large parameter	417
30.1 Notation for uniform asymptotic expansions	417
30.2 The case $a < 0$	419
30.2.1 The case $z > 2\sqrt{-a}, -a + z \rightarrow \infty$	419
30.2.2 The case $z < -2\sqrt{-a}, -a - z \rightarrow \infty$	420
30.2.3 The case $-2\sqrt{-a} < z < 2\sqrt{-a}$	421
30.3 The case $a > 0$	422
30.3.1 The case $z \geq 0, a + z \rightarrow \infty$	423
30.3.2 The case $z \leq 0, a - z \rightarrow \infty$	423
30.4 Expansions from integral representations	424
30.4.1 The case $a > 0, z \geq 0; a + z \rightarrow \infty$	424
30.4.2 The case $a > 0, z \leq 0; a - z \rightarrow \infty$	426
30.4.3 The case $a < 0, z > 2\sqrt{-a}; -a + z \rightarrow \infty$	427
30.5 Airy-type expansions	428
31. Coulomb wave functions	431
31.1 Contour integrals for Coulomb functions	432
31.2 Expansions for $\rho \rightarrow \infty$ and bounded η	433
31.3 Expansions for $\eta \rightarrow \infty$ and bounded ρ	435
31.4 Expansions for $\eta \rightarrow -\infty$ and bounded ρ	437
31.5 Expansions for $\eta \rightarrow -\infty$ and $\rho \geq \rho_0 > 0$	438
31.6 Expansions for $\eta \rightarrow -\infty$ and $\rho \geq 0$	440
31.7 Expansions for $\eta, \rho \rightarrow \infty$; Airy-type expansions	442
32. Laguerre polynomials: Uniform expansions	447
32.1 An expansion for bounded argument	447

32.2	Uniform expansions	449
32.3	An expansion in terms of Airy functions	450
32.4	An expansion in terms of Bessel functions	451
32.5	An expansion in terms of Hermite polynomials	453
33.	Generalized Bessel polynomials	461
33.1	Relations to Bessel and Kummer functions	462
33.2	An expansion in terms of Laguerre polynomials	463
33.3	Expansions in terms of elementary functions	466
33.4	Expansions in terms of modified Bessel functions	472
34.	Stirling numbers	475
34.1	Definitions and integral representations	475
34.2	Stirling number of the second kind	477
34.3	Stirling numbers of the first kind	484
35.	Asymptotics of the integral $\int_0^1 \cos(bx + a/x) dx$	487
35.1	The case $b < a$	487
35.2	The case $a = b$	489
35.3	The case $b > a$	490
35.4	A Fresnel-type expansion	493
A Class of Cumulative Distribution Functions		495
36.	Expansions of a class of cumulative distribution functions	497
36.1	Cumulative distribution functions: A standard form	497
36.2	An incomplete normal distribution function	501
36.3	The Sievert integral	502
36.4	The Pearson type IV distribution	503
36.5	The Von Mises distribution	505
37.	Incomplete gamma functions: Uniform expansions	509
37.1	Using the standard integral representations	509
37.2	Representations by contour integrals	510
37.2.1	Constructing the expansions	512
37.2.2	Details on the coefficients	514
37.2.3	Relations to the coefficients of earlier expansions	516
37.3	Incomplete gamma functions, negative parameters	516
37.3.1	Expansions near the transition point	518
37.3.2	A real expansion of $\gamma^*(-a, -z)$	520

Contents

13

38. Incomplete beta function	521
38.1 A power series expansion for large p	522
38.2 A uniform expansion for large p	522
38.3 The nearly symmetric case	523
38.4 The general error function case	525
39. Non-central chi-square, Marcum functions	527
39.1 Properties of the Marcum functions	528
39.2 More integral representations	529
39.3 Asymptotic expansion; μ fixed, ξ large	531
39.4 Asymptotic expansion; $\xi + \mu$ large	533
39.5 An expansion in terms of the incomplete gamma function	536
39.6 Comparison of the expansions numerically	539
40. A weighted sum of exponentials	541
40.1 An integral representation	542
40.2 Saddle point analysis	543
40.3 Details on the coefficients	544
41. A generalized incomplete gamma function	549
41.1 An expansion in terms of incomplete gamma functions	550
41.2 An expansion in terms of Laguerre polynomials	550
41.3 An expansion in terms of Kummer functions	551
41.4 An expansion in terms of the error function	551
42. Asymptotic inversion of cumulative distribution functions	555
42.1 The asymptotic inversion method	555
42.2 Asymptotic inversion of the gamma distribution	557
42.2.1 Other asymptotic inversion methods	560
42.2.2 On the zeros of incomplete gamma functions	561
42.3 Asymptotic inversion of the incomplete beta function	564
42.3.1 Inverting by using the error function	564
42.3.2 Inverting by using the incomplete gamma function	565
42.4 The hyperbolic cumulative distribution	569
42.5 The Marcum functions	571
42.5.1 Asymptotic inversion	572
42.5.2 Asymptotic inversion with respect to x	572
42.5.3 Asymptotic inversion with respect to y	575
<i>Bibliography</i>	579
<i>Index</i>	593