Game Theory, Maximum Generalized Entropy, Minimum Discrepancy, Robust Bayes and Pythagoras

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Overview

1. Maximum Entropy and Game Theory
2. Maximum Generalized Entropy
3. M.G.E. and Robust Bayes (Result I)
4. Minimum Discrepancy
5. Pythagoras (Result II)
6. Conclusion

Setting

\begin{align*}
\mathcal{X} & \quad \text{Finite (for now) Sample Space} \\
\mathcal{P} & \quad \text{Set of all distributions over } \mathcal{X} \\
\mathcal{C} & \quad \text{Convex Closed Subset of } \mathcal{P} \\
H & \quad \text{Shannon (for now) Entropy:} \\
H(P) := E_P[-\ln P(X)] = -\sum_{x\in\mathcal{X}} P(x) \ln P(x)
\end{align*}

Maximum Entropy Principle

Jaynes 1957

Suppose we only know that \( X \) \( P, P \in \mathcal{C} \)
We are asked to make probabilistic predictions/decisions about \( X \)
According to 'MaxEnt', we should predict using the unique \( \hat{P} \in \mathcal{C} \) that maximizes entropy under the constraint \( \mathcal{C} \):

\[
\hat{P} := \arg \max_{P \in \mathcal{C}} H(P).
\]
Does it make any sense?

- MaxEnt applied in speech recognition, computer vision, stock market prediction...
- …but not always clear why it would be a good idea to use it!
- Various rationales and criticisms have been given over time
- Topsøe (1979) offers a *game-theoretic* interpretation

Basic Result

\[
\text{Information Inequality:} \quad \text{if } P \neq Q \text{ then } E_P[-\ln Q(X)] > E_P[-\ln P(X)] \\
\text{so that} \quad H(P) = \inf_{Q \in \mathcal{P}} E_P[-\ln Q(X)]
\]

\[
H(\hat{P}) = \sup_{P \in \mathcal{C}} H(P) = \sup_{P \in \mathcal{C}} \inf_{Q \in \mathcal{P}} E_P[-\ln Q(X)]
\]

?? Von Neumann 1928 ???
Basic Result, cont.

MaxEnt as a game between Nature and Statistician with loss measured by ‘log loss’ $L(x, Q) := - \ln Q(x)$.

MaxEnt $\bar{P}$ worst-case optimal strategy for Nature:

$$\sup_{P \in \mathcal{C}} H(P) = \sup_{P \in \mathcal{C}} \inf_{Q \in \mathcal{P}} E_P[- \ln Q(X)]$$

achieved for $P = \bar{P}$.

Basic Result, part II

MaxEnt $\bar{P}$ worst-case optimal strategy for Nature:

$$\sup_{P \in \mathcal{C}} H(P) = \sup_{P \in \mathcal{C}} \inf_{Q \in \mathcal{P}} E_P[- \ln Q(X)]$$

achieved for $P = \bar{P}$.

MaxEnt $\bar{P}$ worst-case optimal strategy for Statistician:

$$\inf_{Q \in \mathcal{P}} \sup_{P \in \mathcal{C}} E_P[- \ln Q(X)]$$

achieved for $Q = \bar{P}$.

Basic Result, part II

MaxEnt $\bar{P}$ worst-case optimal strategy for Statistician:

$$\inf_{Q \in \mathcal{P}} \sup_{P \in \mathcal{C}} E_P[- \ln Q(X)]$$

achieved for $Q = \bar{P}$.

Nature has to satisfy constraint

Statistician can choose anything she likes.
Basic Result, part II

MaxEnt $\tilde{P}$ worst-case optimal strategy for Statistician:

$$\inf_{Q \in \mathcal{P}} \sup_{P \in \mathcal{C}} \mathbb{E}_P [- \ln Q(X)]$$

achieved for $Q = \tilde{P}$

- This justifies the Maximum Entropy Principle when the ‘log loss’ is the proper loss function to use:
  – Coding, Kelly Gambling

The Clue

…but what if we are interested in another loss function?

Similar Story Can Still Be Told!

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Game/Decision Theory

$\mathcal{A} \times \mathcal{X} \times \mathcal{C}$ Action Space, Sample Space, Constraint Set

$\mathcal{A}^r$ Randomized actions (set of distributions over $\mathcal{A}$)

$L : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ Loss Function

$L(P, a) := \mathbb{E}_P \mathbb{E}_a[L(X, A)]$

$(\mathcal{C}, \mathcal{A}^r, L)$ Our Game!

Statistician’s Choice

Nature’s Choice
**Example: Logarithmic Loss**

\[ A = P \]

Here actions are formally same as probability distributions

\[ L_{\log}(x, P) := - \ln P(X = x) \quad \text{[} = - \ln p(x) \text{]} \]

Logarithmic loss is a proper scoring rule, i.e. for all \( P \):

\[ P = \arg\min_{\hat{Q} \in A} E_P [- \ln Q(X)] = \arg\min_{\hat{Q} \in A} L_{\log}(P, Q) \]

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**Generalized Entropy**

**CENTRAL DEFINITION**

For (arbitrary) loss function \( L \), the \( L \)-entropy of \( P \) is defined by

\[ H_L(P) := \inf_{a \in A} L(P, a) \]

De Groot 1962
Rao 1982

Shannon Entropy is special case:

\[ H_{\log}(P) = \inf_{Q \in A} L_{\log}(P, Q) = \inf_{Q \in A} E_P [- \ln Q(X)] \]

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**Generalized Entropy**

**CENTRAL DEFINITION**

For (arbitrary) loss function \( L \), the \( L \)-entropy of \( P \) is defined by

\[ H_L(P) := \inf_{a \in A} L(P, a) \]

always concave

(infimum of linear functions)

often differentiable
Example: Brier (squared) Loss

\[ X = \{1, \ldots, k\} \]
\[ A = \mathcal{P} \]
\[ L_{BR}(i, P) := \|\hat{c}_i - \bar{p}\|^2 = \]
\[ (\hat{p}(1))^2 + \cdots + (\hat{p}(i-1))^2 + (1 - \hat{p}(i))^2 + (\hat{p}(i+1))^2 + \cdots + (\hat{p}(k))^2 \]
\[ H_{BR}(P) = \inf_{Q \in A} L_{BR}(P, Q) = L_{BR}(P, P) \]

Brier loss is proper scoring rule

Example: Brier (squared) Loss

\[ X = \{0, 1\} \]
\[ H_{BR}(P) = 2P(1 - P) \]
Example: 0/1 - Loss

\[ X = \{0, 1\} \quad H_{01}(P) = 1 - \sup\{P(0), P(1)\} \]

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Main Theorem (baby version)

Assume
- \( X \) finite
- \( \mathcal{C} \) convex and closed
- \( \mathcal{A} \) closed
- \( L \) bounded from above

...then:

\[ \bar{V} := \sup_{P \in \mathcal{C}} H_L(P) = \sup_{P \in \mathcal{C}} \inf_{a \in \mathcal{A}} L(P, a) \]

is reached for some \( \tilde{P}_L \)

\[ \bar{V} := \inf_{a \in \mathcal{A}'} \sup_{P \in \mathcal{C}} L(\tilde{P}, a) \]

is reached for some \( \tilde{a}_{\tilde{P}_L} \) achieving \( \inf_{a \in \mathcal{A}'} L(\tilde{P}_L, a) \)

\[ \bar{V} = \bar{V} \quad \text{Game has a value!} \]
But what is new here?

- **Mathematically:**
  - Nothing new in baby version
  - In paper we present an adult version
    - General sample spaces, unbounded loss functions, non-compact sets of constraints...
    - New proof technique
- **Conceptually:**
  - ‘maximum generalized entropy is robust Bayes’
  - New view leads to new math results later

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Discrepancy

(= generalized relative entropy)

For given loss function $L$, we can define the discrepancy $D_L(P, a)$ by

$$D_L(P, a) = L(P, a) - \inf_{a \in A} L(P, a)$$

Relative Entropy is special case:

$$D(P || Q) = \sum_x P(x) \ln \frac{P(x)}{Q(x)} = E_P[- \ln Q(X) - (- \ln P(X))]$$

$$= E_P[- \ln Q(X)] - \inf_{Q' \in P} E_P[- \ln Q'(X)].$$

Example Discrepancy: Brier score

$$L_{BR}(x, Q) := ||e_x - q||^2$$

$$L_{BR}(P, Q) = E_{X \sim P} L_{BR}(X, Q)$$

$$D_{BR}(P, Q) = L_{BR}(P, Q) - \inf_{Q' \in P} L_{BR}(P, Q') = ||\bar{p} - \bar{q}||^2 = \sum_x (P(x) - Q(x))^2$$

- This is just the squared Euclidean distance!
Minimum Relative Entropy Principle

For a given 'prior' distribution $Q$ and constraint $C$ pick distribution $\tilde{P}$ achieving

$$\inf_{\tilde{P} \in C} D(P||Q) = \inf_{\tilde{P} \in C} \sum P(X) \ln \frac{P(X)}{Q(X)}$$

- Interpretation: $Q$ is the member of $C$ that is closest to $\tilde{P}$, i.e. it is the projection of $Q$ on $C$

Pythagorean Property

As noted by Csiszár, relative entropy behaves in some ways like squared Euclidean distance:

for all priors $Q$ and all $P \in C$ we have

$$D(P||\tilde{P}) + D(\tilde{P}||Q) \leq D(P||Q)$$

Under some extra conditions we have equality.

Csiszár 1975, 1991, many others

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6. Further Developments
Relative Games

For every loss function $L$ and reference act $e$, we can define the relative loss $L_e(X, a)$ by

$$L_e(X, a) := L(X, a) - L(X, e)$$

Main Theorem

Grünwald and Dawid, 2002

For all $e$, $C$ such that $D_L(P, e)$ is finite for all $P \in C$, the game $(C, A', L_e)$ has a value, i.e.

$$\sup_{P \in C} \inf_{a \in A'} L_e(P, a) = \inf_{a \in A'} \sup_{P \in C} L_e(P, a)$$

reached for saddlepoint $(P_L, \tilde{a}_L)$ if and only if, for all $P \in C$:

$$D_L(P, \tilde{a}_L) + D_L(P_L, e) \leq D_L(P, e)$$

Pythagoras = Von Neumann

Who could have guessed?

In words:
The Pythagorean Property holds iff the minimax theorem applies to the loss function under consideration.

For example:
- minimax theorem holds for squared loss;
- Pythagorean property reduces to high-school Pythagorean theorem.

Conclusion

- We have shown:
  - Maximum (Gen.) Entropy = Robust Bayes
  - Pythagoras = Von Neumann
- Three further results in full paper:
  - Relation to Bregman divergences
  - Generalized Exponential Families
  - Generalized Redundancy-Capacity (Gallagher-Ryabko-Haussler) Theorem
Thank you for your attention!

How general is Pythagorean property?

- Both squared Euclidean distance and relative entropy are examples of Bregman divergences
- Pythagoras known to hold for such divergences