Occam, Bayes, MDL and the Real World!

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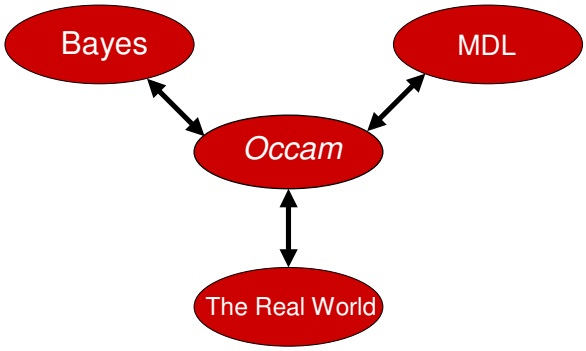
Three Mysteries



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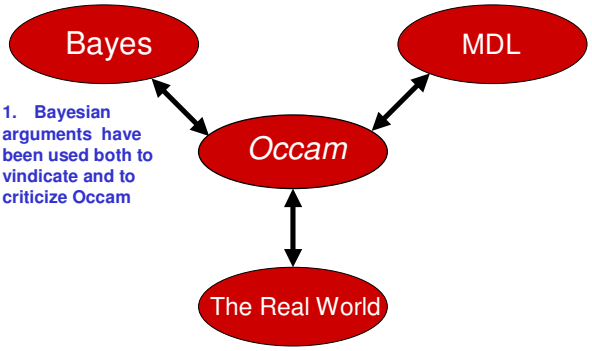

1. Bayesian arguments have been used both to vindicate and to criticize Occam

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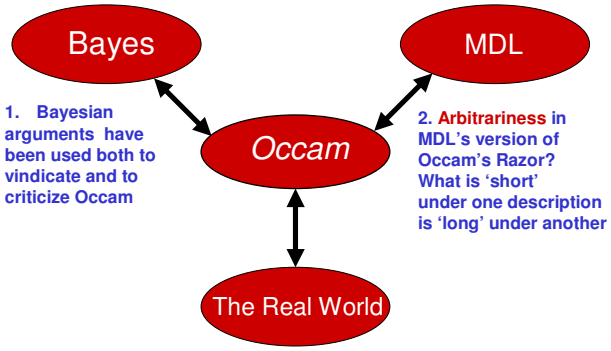

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2. Arbitrariness in MDL's version of Occam's Razor? What is 'short' under one description is 'long' under another

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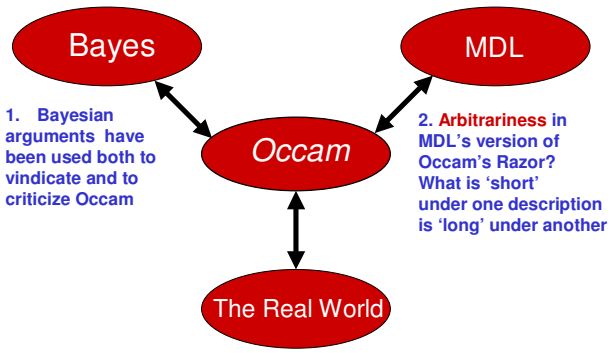
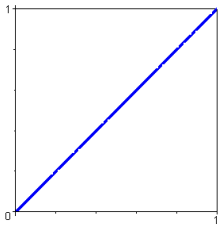


1. Bayesian arguments have been used both to vindicate and to criticize Occam

2. **Arbitrariness** in MDL's version of Occam's Razor? What is 'short' under one description is 'long' under another

3. How can a preference for 'simplicity' make sense in a complex world?

Bayes and Occam

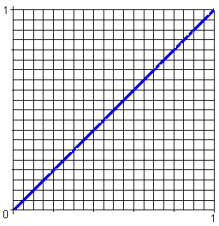


$x^n = x_1, \ldots, x_n$ Binary sequence of data

$\mathcal{M}_1$ Class (model) of i.i.d. Bernoulli distributions

$\mathcal{M}_2$ Class (model) of First-Order Markov Chains

Bayes and Occam



$\ddot{\mathcal{M}}_1$ **Discretized** i.i.d. Bernoulli distributions

$\ddot{\mathcal{M}}_2$ **Discretized** First-Order Markov Chains

Bayesian justification of Occam

'Occam Factor'-type Argument (Gull '88)

- No prior preference for $\ddot{\mathcal{M}}_1$ or $\ddot{\mathcal{M}}_2$.
  - expressed as $P(\ddot{\mathcal{M}}_1) = P(\ddot{\mathcal{M}}_2) = \frac{1}{2}$
- *Given* $\ddot{\mathcal{M}}_j$, no preference for any of the distributions in $\ddot{\mathcal{M}}_j$:
  - i.e. for all θ indexing a distribution in $\ddot{\mathcal{M}}_j$, $P(\theta|\ddot{\mathcal{M}}_j) = \text{const.} = \frac{1}{|\ddot{\mathcal{M}}_j|}$.
  - for example: $P(\theta|\ddot{\mathcal{M}}_1) = \frac{1}{100}$; $P(\theta|\ddot{\mathcal{M}}_2) = \frac{1}{100 \times 100} = \frac{1}{10000}$

Bayesian justification of Occam

'Occam Factor'-type Argument (Gull '88)

- Bayesian model selection selects

$$\arg\max_j P(\ddot{\mathcal{M}}_j|x^n) = \arg\max_j P(x^n|\ddot{\mathcal{M}}_j)P(\ddot{\mathcal{M}}_j) =$$

$$\arg\max_j \sum_{\theta:P(\cdot|\theta)\in\ddot{\mathcal{M}}_j} P(x^n|\theta)P(\theta|\ddot{\mathcal{M}}_j)$$

a lot smaller for $\ddot{\mathcal{M}}_2$

Bayesian justification of Occam

'Occam Factor'-type Argument (Gull '88)

Prior for individual distribution *within* 'complex' model is much smaller. Therefore, if the simple and the complex model fit the data about equally well, Bayes selects 'simple' model.

Bayesian criticism of Occam

'No Free Lunch'-type Argument (Van Allen, Greiner '00)

- No prior preference for $\ddot{\mathcal{M}}_1$ or $\ddot{\mathcal{M}}_2$.
  - previously expressed as
    $$P(\ddot{\mathcal{M}}_1) = P(\ddot{\mathcal{M}}_2) = \tfrac{1}{2}$$
  - now expressed as
    $$P'(\theta) = \text{const.} = \frac{1}{|\ddot{\mathcal{M}}_1 \cup \ddot{\mathcal{M}}_2|} = \frac{1}{|\ddot{\mathcal{M}}_2|}.$$
  - for example:
    $$P'(\ddot{\mathcal{M}}_1) = \textstyle\sum_{\theta\,:\,P(\cdot|\theta)\in\ddot{\mathcal{M}}_1} P'(\theta) = 100 \times \frac{1}{10000} = \frac{1}{100}$$
    $$P'(\ddot{\mathcal{M}}_2) = \textstyle\sum_{\theta\,:\,P(\cdot|\theta)\in\ddot{\mathcal{M}}_2\setminus\ddot{\mathcal{M}}_1} P'(\theta) = 1 - \frac{1}{100} = \frac{99}{100}$$

Bayesian criticism of Occam

- As discretization gets finer and finer, $\ddot{\mathcal{M}}_1$ gets swamped by $\ddot{\mathcal{M}}_2$ in the sense that
  $$\frac{P'(\ddot{\mathcal{M}}_1)}{P'(\ddot{\mathcal{M}}_2)} \to 0$$
- Therefore, with prior $P'(\ddot{\mathcal{M}}_j)$, Bayesian model selection will always select $\ddot{\mathcal{M}}_2$, no matter what data/sample size we actually observe!

Bayesian criticism of Occam

'No Free Lunch'-type Argument (Van Allen, Greiner '00)

- No prior preference for $\ddot{\mathcal{M}}_1$ or $\ddot{\mathcal{M}}_2$.
  - previously expressed as
    uniform prior over things you are interested in
  - now expressed as
    uniform prior over possible states of the world

Who's Right??

short answer:

The validity of either argument depends entirely on what you mean by *'Bayesian Statistics'*!

Savage, De Finetti, Jeffreys

'modern' Bayesian Statistics has (at least) three founding fathers, each with (quite) different ideas

- **L. Savage**
  The Foundations of Statistics (1954)
- **B. De Finetti**
  Theory of Probability ('1937',1974)
- **H. Jeffreys**
  Theory of Probability (1939, 1961)

Subjective vs Pragmatic Priors

- **Savage**
  - most influential of the three
  - $P(\theta)$ is quite literally 'degree of belief that θ is true'
- **De Finetti**
  - Allows *pragmatic* priors
  - $P(\theta)$ *cannot* be interpreted as 'degree of belief that θ is true' (nevertheless, subjectivist)

'Probabilities do not exist'

B. De Finetti, 1974, page 1

Purely Subjective vs Pragmatic Priors

- If you insist on Savage's interpretation, and you believe that the distributions in $\ddot{\mathcal{M}}_1$ are not a priori more likely than those in $\ddot{\mathcal{M}}_2$, then you end up with NFL-type argument
- If you accept De Finetti/Jeffreys, you may choose to use Occam-type prior if it is *useful*.

Purely Subjective vs Pragmatic Priors

- If you insist on Savage's interpretation, and you believe that the distributions in $\ddot{\mathcal{M}}_1$ are not a priori more likely than those in $\ddot{\mathcal{M}}_2$, then you end up with NFL-type argument
  - IMHO, Savage's interpretation is untenable when viewed as `sole valid interpretation' of Bayesian inference: naïve Bayes, speech recognition…
- If you accept De Finetti/Jeffreys, you may choose to use Occam-type prior if it is *useful*.

So, are Occam-type priors useful?

So, are Occam-type priors useful?

YES!

Occam-type priors *are* useful

- **Empirical** justifications:
  - very good results for regression, Bayesian network order selection, denoising…
- **Theoretical** justifications:
  - leads to *consistent* model selection procedures
  - avoid multiple hypothesis testing:
    - **Predictive ('prequential') interpretation**

Prequential Interpretation

Dawid 1984, Rissanen 1984

- For data $x_1, \ldots, x_n$, Bayes with Occam-type prior selects $\ddot{\mathcal{M}}_j$ minimizing

$$\sum_{i=1}^{n} \mathrm{loss}(x_i, P_{\mathrm{preq}}(\cdot|x_1, \ldots, x_{i-1}, \mathcal{M}_j))$$

where

$$P_{\mathrm{preq}}(X_i|x_1, \ldots, x_{i-1}, \mathcal{M}_j) = \int P(X_i|\theta, \mathcal{M}_j) w(\theta|x_1, \ldots, x_{i-1}, \mathcal{M}_j) d\theta$$

$$\mathrm{loss}(x, P) := -\log P(x)$$

- In words: Bayesian model selection selects the model such that Bayesian prediction based on the model leads to the smallest sequential accumulated prediction error, measured using log-loss
- Closely related to **cross-validation**!

Prequential Interpretation

- This suggests, and for some many types of models experiments confirm, that Occam-Bayes selects the model that *leads to smaller prediction error of future data!*
  - For small sample size, this is with high probability the simpler model, even if the 'truth', generating the data is complex!
  - Of course, we have to assume *some* things for this to be true.

Prequential Justification

- Prequential interpretation gives a non-asymptotic justification of Occam-type priors:

If the goal is to minimize prediction error over future data, then selecting an overly simple model may be a good idea even if the truth is complex!

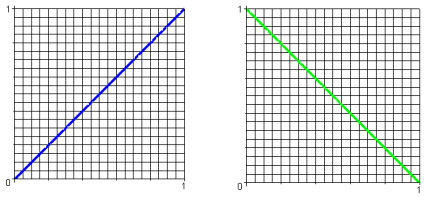
- Closely related to bias-variance trade-off, cross-validaton

Occam-type priors and MDL

- with Occam-type priors, Bayesian model selection becomes very similar to 'modern' versions of MDL…
- … but not the same (tutorial tomorrow!)
- gives alternate justification for MDL
- yet one nagging problem remains:
  - Possible *arbitrariness* in definition of 'Occam-type prior' (and hence MDL…)

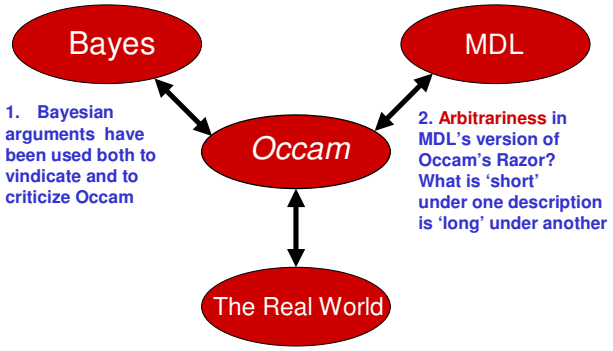
Arbitrariness?

Given $\ddot{\mathcal{M}}_2$, how should we construct $\ddot{\mathcal{M}}_1$?



Why is Bernoulli (left) more natural candidate for $\ddot{\mathcal{M}}_1$ than `reverse Bernoulli' (right) or *any other 1-dimensional submodel* of $\ddot{\mathcal{M}}_2$, for that matter?

One Mystery to Go



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Simple models in a complex world

- Remark:
  Occam's Razor seems no good, because, after all, `What good are simple models in a complex world?'
  G. Webb (as **quoted** in KDD Nuggets 96:2)

- Answer:
  Occam's Razor is useful after all, because it is `mostly true in most real world situations'
  G. Piatetski-Shapiro (KDD Nuggets 96:2)
  (Piatetski later retracted this statement)
  (thanks to Pedro Domingos for telling me this)

'truth' of Occam's Razor is not the point!

- MDL and Bayes with pragmatic priors are *strategies* for inductive inference …
  - **Strategies are not 'true' or 'false', but 'clever' or 'stupid'!**
- …these strategies are *not at all* based on belief that 'simple models are a priori more likely to be true'
  - that idea derives from (untenable yet very influential) purely Savagian interpretation of Bayesian inference
  - much work on MDL based on assumption that 'truth is infinitely complex' (Barron and Cover, 1991)

Simple models in a complex world

- A preference for simplicity can lead to algorithms achieving better predictions for small samples, *even if truth is complex*
  - Of course *some* regularity conditions are needed!
  - Criticisms usually mention boosting, decision trees. *These are very special (yet interesting) cases!*

Thank you for your attention!