MDL exercises, ninth handout

(due May 2nd)

1. Let $f(x)$ be a density function on $[0, \infty)$ with fixed mean $1/\lambda$. Define $g(x) = \lambda e^{-\lambda x}$, the density function of the exponential distribution on the same domain and with the same mean. Show that $H(f)$ is maximized by choosing $f = g$, by evaluating $0 \leq D(f \| g)$.

2. Show that for an exponential family (in the natural parameterization) we have $I(\beta) = \mathrm{Var}(\phi(X))$. It is easiest to use $I(\beta) = E_\beta[(\frac{d}{d\beta} \ln P(X \mid \beta))^2]$.

3. Jensen's inequality states that $E[f(X)] \geq f(E[X])$ for conve^x f. Use this inequality to find (a) a lower bound on the entropy $H(P)$ for a distribution P on a finite sample space, and (b) an upper bound on this entropy (Hint: for the upper bound, rewrite the entropy as $-\sum_x P(x)(f(1/P(x)))$ with $f \equiv -\log$). Compare this upper bound to the entropy for the uniform distribution on that sample space and for a nonuniform distribution on that space. In which case is the bound tighter?

4. For two distributions P_0 and P_1 defined on the same space $\mathcal{X}$ with $P_0 \neq P_1$, let P_α be the α- mixture between P_0 and P_1, i.e. $P_\alpha(x) = (1 - \alpha)P_0(x) + \alpha P_1(x)$. Show that the entropy $H(P_\alpha)$ is strictly concave as a function of $\alpha \in [0, 1]$.