Combinatorics with Computer Science Applications
Final exam

Wed May 30, 2012, 13.00–16.00, UvA Science Park, Room G4.15

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1. **(1 point)** Let n be a positive even integer, and G be a connected 2-regular graph which is a $(n, 2, c)$-expander. Show that $c \leq 4/n$.

2. (a) **(1 point)** Let n, m, and d be positive integers. Suppose we have $2n$ binary vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$ of dimension m (i.e., $u_i \in \{0,1\}^m$ and $v_i \in \{0,1\}^m$ for every $i \in [n]$) with the following inner-product properties over the field of real numbers:

   $\langle u_i, v_i \rangle = 0$ for every $i \in [n]$
   $\langle u_i, v_j \rangle \in \{1, \ldots, d\}$ for every $i \neq j$

   Prove that $n \leq \sum_{i=0}^{d} \binom{m}{i}$.

   (b) **(1 point)** Suppose we have $2n$ subsets $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ of an m-element universe with the following intersection properties:

   $A_i \cap B_i = \emptyset$ for every $i \in [n]$
   $|A_i \cap B_j| \in \{1, \ldots, d\}$ for every $i \neq j$

   Prove that $n \leq \sum_{i=0}^{d} \binom{m}{i}$.

3. **(2 points)** Let n and m be positive integers. Let $G = (V, E)$ be an undirected graph on $|V| = n$ vertices with $|E| = m$ edges. A *bipartition* of the graph divides V into two disjoint sets V_1 and V_2 (with $V = V_1 \cup V_2$). Prove that there is a bipartition of G with at least $m/2$ edges going from V_1-vertices to V_2-vertices.

4. Consider an undirected bipartite graph $G = (V_1 \cup V_2, E)$, where V_1 and V_2 are disjoint sets of n vertices each, and $E \subseteq V_1 \times V_2$ is the set of edges between V_1-vertices and V_2-vertices. The *adjacency matrix* of G is the following $n \times n$ matrix A of 0s and 1s: rows are indexed by V_1, columns by V_2, and the (i, j)-entry $A_{i,j}$ is 1 iff $(i, j) \in E$.

   (a) **(1 point)** Suppose we take the matrix A and replace each non-zero entry $A_{i,j}$ by a variable $x_{i,j}$ (the 0-entries remain 0). Call the resulting matrix $A(x)$, where x is the sequence of all variables. The determinant[1] of $A(x)$ is a polynomial in those variables, of total degree at most n. Show that this polynomial is identically equal to 0 iff the graph G does *not* have a perfect matching.

(b) **(1 point)** Give a randomized algorithm to decide (with error probability less than 1/3) whether G has a perfect matching, using $O(n^{2.38})$ computational steps.
NB: you're not required to *find* a perfect matching, only to decide if one exists.

5. **(2 points)** In the lecture we saw a probabilistic communication protocol for the equality function on inputs $(x, y) \in \{0,1\}^n \times \{0,1\}^n$ that used *private* coin flips and $O(\log n)$ bits of communication. Give a protocol that computes the equality function with error probability $\leq 1/3$ on all possible inputs $(x, y) \in \{0,1\}^n \times \{0,1\}^n$, using *public* coin flips and $O(1)$ bits of communication.
NB: The number of public coin flips is not counted as part of the communication, so you can use as many as you want.

[1]The *determinant* of an $n \times n$ matrix A is defined as

$$\sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} A_{i,\pi(i)},$$

where the sum is over the set S_n of all $n!$ permutations on n elements, and $\mathrm{sgn}(\pi) \in \{+1, -1\}$ is the signature of the permutation π (it doesn't matter if you don't know what that is). It is known that we can compute the determinant of an $n \times n$ matrix in $O(n^\omega)$ computational steps, for some $\omega \in [2, 2.38]$. Here an addition or multiplication in the field $\mathbb{F}$ counts as one computational step. This ω is the *matrix multiplication exponent*; its precise value is unknown, though it is conjectured to equal 2.