Combinatorics with Computer Science Applications
Midterm exam

Thu March 29, 2012, 13.00–16.00, UvA Science Park, Room G2.13
Ronald de Wolf, rdewolf@cwi.nl

1. **(1 point)** Let n be a nonnegative integer. Give two proofs of the identity $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, one algebraic proof and one combinatorial proof.

2. **(1 point)** Let $\mathcal{F}$ be a k-uniform family of sets over $[n]$. The *shadow* $\mathcal{S}$ of $\mathcal{F}$ is the set of all $(k-1)$-element sets that lie entirely in at least one member of $\mathcal{F}$, i.e., $\mathcal{S} = \{A \subseteq [n] : |A| = k-1 \text{ and } \exists B \in \mathcal{F} \text{ s.t. } A \subset B\}$. Show that the size of $\mathcal{S}$ is at least $k|\mathcal{F}|/(n-k+1)$ elements.

3. **(2 points)** Derive the Erdős-Szekeres theorem (Theorem 4.5 in the book) from Theorem 8.1 of the book.

4. **(2 points)** Let $G = (V, E)$ be an undirected graph without self-loops. A *matching* of size k in G is a set of k vertex-disjoint edges. A *star* of size k in G is a set of k edges that all share a vertex a (i.e., edges $(a, b_1), (a, b_2), \ldots, (a, b_k)$). Prove that every graph with more than $2(k-1)^2$ edges contains a matching of size k or a star of size k.

   Hint: view the edges of the graph as a 2-uniform family of sets.

5. **(3 points)** Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. For $x \in \{0,1\}^n$ and $S \subseteq [n]$, let x^S denote x after changing all bits x_i with $i \in S$. For example, $0101^{\{2,3\}} = 0011$. For $i \in [n]$ abbreviate $x^i = x^{\{i\}}$. The *sensitivity* of f at input x is $s(f, x) = |\{i : f(x) \neq f(x^i)\}|$. The sensitivity of f is $s(f) = \max_{x \in \{0,1\}^n} s(f, x)$. The *block sensitivity* of f at input x is $bs(f, x)$, which is the largest number k such that there exist disjoint sets $S_1, \ldots, S_k$ satisfying $f(x) \neq f(x^{S_i})$ for all $i \in [k]$. The block sensitivity of f is $bs(f) = \max_{x \in \{0,1\}^n} bs(f, x)$.

   (a) Show that $s(f) \leq bs(f)$.

   (b) Show that if $S \subseteq [n]$ is a *minimal* set satisfying $f(x) \neq f(x^S)$, then $|S| \leq s(f)$.

   (c) Show that if $C : S \to \{0,1\}$ is a certificate for input x (Section 9.4 of the book), then S intersects each set T for which $f(x) \neq f(x^T)$.

   (d) Show that $C_1(f) \leq bs(f) \cdot s(f)$, where $C_1(f) = \max_{x: f(x)=1} C(f, x)$.

   (e) Show that the decision tree complexity of f is at most $bs(f)^4$.