

Quantum Zero-Error Algorithms Cannot be Composed*

Harry Buhrman
CWI and U. of Amsterdam
buhrman@cwi.nl

Ronald de Wolf
CWI
rdewolf@cwi.nl

Abstract

We exhibit two black-box problems, both of which have an efficient quantum algorithm with zero-error, yet whose composition does not have an efficient quantum algorithm with zero-error. This shows that quantum zero-error algorithms cannot be composed. In oracle terms, we give a relativized world where $ZQP^{ZQP} \neq ZQP$, while classically we always have $ZPP^{ZPP} = ZPP$.

Keywords: Analysis of algorithms. Quantum computing. Zero-error computation.

1 Introduction

We can define a “zero-error” algorithm of complexity T in two different but essentially equivalent ways: either as an algorithm that always outputs the correct value with *expected* complexity T (expectation taken over the internal randomness of the algorithm), or as an algorithm that outputs the correct value with probability at least $1/2$, never outputs an incorrect value, and runs in *worst-case* complexity T . Expectation is linear, so we can compose two classical algorithms that have an efficient expected complexity to get another algorithm with efficient expected complexity. If algorithm A uses an expected number of a applications of B and an expected number of a' other operations, then using a subroutine for B that has an expected number of b operations gives A an expected number of $a \cdot b + a'$ operations. In terms of complexity classes, we have

$$ZPP^{ZPP} = ZPP,$$

where ZPP is the class of problems that can be solved by a polynomial-time classical zero-error algorithm. This equality clearly relativizes, i.e., it holds relative to any oracle A .

In this paper we show that this seemingly obvious composition fact does *not* hold in the quantum world. We exhibit black-box (query complexity) problems g and h that are both easy to quantum compute in the expected sense, yet whose composition $f = g(h, \dots, h)$ requires a very large expected number of queries. In complexity terms, we exhibit an oracle A where

$$ZQP^{ZQP^A} \neq ZQP^A,$$

where ZQP is the class of problems that can be solved by a polynomial-time quantum zero-error algorithm. This result is somewhat surprising, because *exact* quantum algorithms can easily be composed, and so can *bounded-error* quantum algorithms. Moreover, it is also easy to use a quantum zero-error algorithm as a subroutine in a *classical* zero-error algorithm. That is

$$EQP^{EQP} = EQP \text{ and } BQP^{BQP} = BQP \text{ and } ZPP^{ZQP} = ZQP,$$

relativized as well as unrelativized.

*Partially funded by projects QAIP (IST-1999-11234) and RESQ (IST-2001-37559) of the IST-FET programme of the EC.

2 Preliminaries

We assume familiarity with computational complexity theory [9] and quantum computing [8]. In this section we briefly introduce the “modes of computation” that we are considering. Let f be some (possibly partial) Boolean function with set of inputs $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$, where $f(\mathcal{X}_0) = 0$ and $f(\mathcal{X}_1) = 1$. Let $P_b(x)$ be the probability that algorithm A outputs bit b on input x . We define four modes of computation:

1. A is an *exact* algorithm for f if $P_1(x) = 1$ for all $x \in \mathcal{X}_1$ and $P_0(x) = 1$ for all $x \in \mathcal{X}_0$
2. A is a *zero-error* algorithm for f if $P_1(x) \geq 1/2$ and $P_0(x) = 0$ for all $x \in \mathcal{X}_1$ (assume there is a third possible output “don’t know”), and $P_0(x) \geq 1/2$ and $P_1(x) = 0$ for all $x \in \mathcal{X}_0$
3. A is a *bounded-error* algorithm for f if $P_1(x) \geq 2/3$ for all $x \in \mathcal{X}_1$, and $P_0(x) \geq 2/3$ for all $x \in \mathcal{X}_0$
4. A is a *nondeterministic* algorithm for f if $P_1(x) > 0$ for all $x \in \mathcal{X}_1$, and $P_1(x) = 0$ for all $x \in \mathcal{X}_0$

Note that an exact algorithm is a zero-error algorithm, and a zero-error algorithm is a bounded-error algorithm as well as a non-deterministic algorithm.

In the setting of *query complexity*, f is an N -bit Boolean function, so $\mathcal{X}_0 \cup \mathcal{X}_1 \subseteq \{0, 1\}^N$. We can only access the input $x \in \{0, 1\}^N$ by making queries to its bits. A query is the application of the unitary transformation O_x that maps

$$O_x : |i, b, z\rangle \mapsto |i, b \oplus x_i, z\rangle,$$

where $i \in [N]$ and $b \in \{0, 1\}$. The z -part corresponds to the workspace, which is not affected by the query. A T -query quantum algorithm has the form $A = U_T O_x U_{T-1} \cdots O_x U_1 O_x U_0$, where the U_k are fixed unitary transformations independent of x . The final state $A|0\rangle$ depends on x via the T applications of O_x . The output of the algorithm is determined by measuring the two rightmost qubits of the final state. Let’s say that if the rightmost bit is 1 then the algorithm claims ignorance (“don’t know”), and if it is 0 then the next-to-rightmost bit is the output bit. We refer to the survey [3] for more details about classical and quantum query complexity.

We will use $Q_E(f)$, $Q_0(f)$, $Q_2(f)$, $NQ(f)$ to denote the minimal query complexity of a quantum algorithm for f in the four above modes, respectively. Accordingly, $Q_E(f)$ is the exact quantum query complexity of f , $Q_0(f)$ is zero-error quantum query complexity, $Q_2(f)$ is bounded-error quantum query complexity, and $NQ(f)$ is nondeterministic quantum query complexity. Note that by definition we immediately have

$$Q_2(f) \leq Q_0(f) \leq Q_E(f) \quad \text{and} \quad NQ(f) \leq Q_0(f) \leq Q_E(f).$$

Our proofs will use the close connection between quantum query complexity and polynomials [2]. An N -variate multilinear polynomial p is a function of the form $p(x) = \sum_{S \subseteq [N]} a_S x_S$, where a_S is real and $x_S = \prod_{i \in S} x_i$. Its degree $\deg(p) = \max\{|S| : a_S \neq 0\}$ is the largest degree among its monomials. The next lemma [6, 11] connects nondeterministic complexity with polynomials:

Lemma 1 *The nondeterministic quantum query complexity $NQ(f)$ of f equals the minimal degree among all multilinear polynomials p such that*

1. $p(x) \neq 0$ for all $x \in \mathcal{X}_1$

2. $p(x) = 0$ for all $x \in \mathcal{X}_0$

This lemma improves the query complexity lower bound by a factor of 2, compared to the “standard” polynomial method [2].

The setting of computational complexity can be defined either in terms of Turing machines or of uniform circuit families. Here we define EQP, ZQP, BQP, and NQP to be the classes of languages for which there exist polynomial-time quantum algorithms in the above four modes, respectively. We restrict attention to algebraic amplitudes for these classes.

For example, NQP (“quantum NP”) is taken to be the class of languages L for which there exists an efficient quantum algorithm that has positive acceptance probability on input x iff $x \in L$ [1]. This class was shown to be equal to the classical counting class coC=P [5, 12]. There is an alternative definition of quantum NP based on verification of quantum certificates [7, Chapter 14] which we will not discuss here. We similarly define the classes EQP^A , etc., when we have access to an oracle A for some language, and $\text{EQP}^S = \cup_{A \in S} \text{EQP}^A$, etc., when S is a set of oracles. By definition we immediately have

$$\text{EQP} \subseteq \text{ZQP} \subseteq \text{BQP} \quad \text{and} \quad \text{EQP} \subseteq \text{ZQP} \subseteq \text{NQP},$$

and these inclusions also hold relative to any oracle A .

3 The problem

Let m and n be even numbers. We first define the partial Boolean functions g on n bits and h on $2m$ bits, and then their composition f on $N = 2mn$ bits.

The function g is just the constant vs. balanced problem of Deutsch and Jozsa [4]. Using $w(x)$ to denote the Hamming weight of $x \in \{0, 1\}^n$, we define:

$$g(x) = \begin{cases} 1, & \text{if } w(x) = 0 \quad (\text{constant}) \\ 0, & \text{if } w(x) = n/2 \quad (\text{balanced}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is well known that there exists an exact 1-query quantum algorithm for this problem [4], while any classical deterministic or even zero-error algorithm needs $n/2 + 1$ queries.

The function h is a zero-error sampling problem. Let

$$\begin{aligned} \mathcal{A}_1 &= \{0^m x : x \in \{0, 1\}^m, m/2 \leq w(x) \leq m\} \\ \mathcal{A}_0 &= \{x 0^m : x \in \{0, 1\}^m, m/2 \leq w(x) \leq m\} \\ h(x) &= \begin{cases} 1, & \text{if } x \in \mathcal{A}_1 \\ 0, & \text{if } x \in \mathcal{A}_0 \\ \text{undefined} & \text{otherwise} \end{cases} \end{aligned}$$

Clearly h has a classical algorithm that always outputs the correct answer and whose expected number of queries is small. The algorithm just queries a random point in the first m bits of its input and one in the second m bits, and outputs where it finds a 1 (if it does so). With probability $\geq 1/2$ it will indeed find a 1, so the expected number of repetitions before termination is ≤ 2 .

Let f on $2mn$ bits be the partial Boolean function that is the composition of g and h . In other words, defining the set of promise inputs by

$$\begin{aligned} \mathcal{X}_1 &= \underbrace{\mathcal{A}_0 \times \cdots \times \mathcal{A}_0}_{n \text{ times}} \\ \mathcal{X}_0 &= \cup \{ \mathcal{A}_{y_1} \times \cdots \times \mathcal{A}_{y_n} : y = y_1 \dots y_n \in \{0, 1\}^n, w(y) = n/2 \} \end{aligned}$$

we have

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathcal{X}_1 \text{ (constant)} \\ 0, & \text{if } x \in \mathcal{X}_0 \text{ (balanced)} \\ \text{undefined} & \text{otherwise} \end{cases}$$

For later reference, we will give names to the various parts of the $2mn$ -bit input x :

$$x = \overbrace{\underbrace{x^{(0,1)} x^{(1,1)}}_{\substack{\text{input for } h \\ \text{mbits} \quad \text{mbits}}} \underbrace{x^{(0,2)} x^{(1,2)}}_{\substack{\text{input for } h \\ \text{mbits} \quad \text{mbits}}} \dots \dots \dots \underbrace{x^{(0,n)} x^{(1,n)}}_{\substack{\text{input for } h \\ \text{mbits} \quad \text{mbits}}}}^{\text{input for } g}$$

In words, f contains n different h -functions, each with its own $2m$ -bit input. Here $x^{(0,i)}$ and $x^{(1,i)}$ are two m -bit strings that together constitute the input to the i th h -function. The promise says that the $2m$ -bit input $x^{(0,i)}x^{(1,i)}$ always lies in \mathcal{A}_0 or \mathcal{A}_1 . The n bits $h(x^{(0,i)}x^{(1,i)})$, $i = 1, \dots, n$, coming out of the n h -functions are then plugged into g to give the value for f . The promise says that these n bits are either all 0 (constant) or half 0 and half 1 (balanced).

Our function f is just the composition of the problems g and h , each of which needs just a small expected number of queries. Yet below we will show that any quantum zero-error algorithm for f will need to make *many* queries. Even stronger, also a *nondeterministic* quantum algorithm for f requires many queries.

4 Lower bound for quantum zero-error algorithms

The next lemma is our main technical tool:

Lemma 2 *Let p be a $2mn$ -variate multilinear polynomial such that*

1. $p(x) \neq 0$ for all $x \in \mathcal{X}_1$
2. $p(x) = 0$ for all $x \in \mathcal{X}_0$

Then $\deg(p) \geq \min(n/2, m/2) + 1$.

Proof. We use the names for the various subparts of the $2mn$ -bit input that we introduced in Section 3. We assume without loss of generality that for every $i \in [n]$ and every non-zero monomial $a_S x_S$ in p , the set S does not simultaneously contain variables from $x^{(0,i)}$ and from $x^{(1,i)}$. Since the promise on the inputs sets either $x^{(0,i)}$ or $x^{(1,i)}$ to 0^m , a monomial containing variables from both $x^{(0,i)}$ and $x^{(1,i)}$ evaluates to 0 anyway, so removing it from p will not affect the two properties of p .

Suppose, by way of contradiction, that $d = \deg(p) \leq \min(n/2, m/2)$. By the first property of the lemma, p cannot be identically zero, so it has to contain at least one monomial. Consider a monomial $M = a_S x_S$ in p with maximal degree, so $|S| = d$. Consider some $i \in [n]$ such that S contains variables from $x^{(1,i)}$ (and hence, by the above assumption, no variables from $x^{(0,i)}$). We now fix $x^{(0,i)}$ to 0^m and fix all non- S variables in $x^{(1,i)}$ to 1. Since there are at most $m/2$ S -variables in total, this already sets at least $m/2$ bits in $x^{(1,i)}$ to 1. Accordingly, we have $x^{(0,i)}x^{(1,i)} \in \mathcal{A}_1$ for every setting of the S -variables. This forces the i th h -function to value 1, without fixing the S -variables. Similarly we force the other h -functions whose variables intersect with S : if S has variables from $x^{(1,j)}$ then we force the j th h -function to 1, and if S has variables from $x^{(0,j)}$ then we force it to 0. Since $|S| \leq n/2$, this forces at most $n/2$ of the h -functions. Accordingly, we can

extend our setting to the other h -functions (whose variables don't intersect with S at all) to create a setting of the overall $2mn$ -bit input that is in \mathcal{X}_0 (balanced), without fixing the S -variables.

Let q be the remaining polynomial in the d S -variables. No matter how we vary the S -variables, the overall input to p remains in \mathcal{X}_0 (balanced). Hence q must be zero on all Boolean settings of its variables. It is easy to see that the only polynomial satisfying this constraint is the one without any monomials. But q still contains the monomial M , because being of degree d , M cannot cancel against other monomials when we fix the non- S variables. This is a contradiction. \square

This lemma is exactly tight. First, there is a polynomial with the above properties of degree $n/2 + 1$. For T a set of $n/2 + 1$ variables, each from a different $x^{(0,i)}$, define q_T to be the degree- $(n/2 + 1)$ polynomial that is the AND of these variables. If $x \in \mathcal{X}_0$ then q_T will be 0 for all T , and if $x \in \mathcal{X}_1$ then for at least one T we have $q_T = 1$. Hence summing q_T over all such T gives a polynomial p of degree $n/2 + 1$ such that $p(x) = 0$ for $x \in \mathcal{X}_0$ and $p(x) > 0$ for $x \in \mathcal{X}_1$.

Second, there also is an appropriate polynomial of degree $m/2 + 1$. Let q_i be the degree- $(m/2 + 1)$ polynomial that is the OR of the first $m/2 + 1$ bits of $x^{(1,i)}$. Then $q_i = 1$ if $x^{(0,i)}x^{(1,i)} \in \mathcal{A}_1$ and $q_i = 0$ if $x^{(0,i)}x^{(1,i)} \in \mathcal{A}_0$. Defining p to be the degree- $(m/2 + 1)$ polynomial $n/2 - \sum_{i=1}^n q_i$, we have $p(x) = 0$ for $x \in \mathcal{X}_0$ and $p(x) = n/2$ for $x \in \mathcal{X}_1$.

Combining the previous lemma with Lemma 1 gives our main theorem:

Theorem 1 $NQ(f) = \min(n/2, m/2) + 1$.

Since nondeterministic query complexity lower bounds zero-error complexity, we also obtain the zero-error lower bound $Q_0(f) \geq \min(n/2, m/2) + 1$. The best upper bound on $Q_0(f)$ that we know, is $\min(2n, m)$ so the lower bound is tight up to small constant factors. First, we know there is a classical zero-error algorithm that computes an h -function using an expected number of 2 queries; we can use this to compute the first $n/2$ h -functions in an expected number of n queries, which suffices to compute f . Terminating this algorithm after $2n$ steps gives us an algorithm that finds the correct output with probability $\geq 1/2$ (Markov's inequality), and claims ignorance otherwise.

Second, there exists an exact quantum algorithm for f that uses m queries. By querying the first $m/2$ bits in an h -input we can decide whether that h takes value 0 or 1. By copying the output and reversing the computation we can do this exact computation cleanly (resetting all workspace to 0) using m queries. Putting the Deutsch-Jozsa algorithm on top of this gives an m -query exact quantum algorithm for f .

Using a standard translation of query complexity results to oracles, we obtain

Theorem 2 *There exists an oracle A such that*

$$\text{EQP}^{\text{ZPP}^A} \not\subseteq \text{NQP}^A,$$

hence in particular

$$\text{ZQP}^{\text{ZQP}^A} \not\subseteq \text{ZQP}^A.$$

Proof. For a set $A \subseteq \{0, 1\}^*$, we use $A^{=n}$ to denote the set of all n -bit strings in A , and we identify this with its 2^n -bit characteristic vector. We will construct a set A such that, for every n where $2^n = 2m^2$ for some m (i.e. for every odd n), $A^{=n}$ is a valid input to f (word of warning: the ' n ' used here is not the ' n ' used earlier, but the ' m ' is; the input length of f is now $2m^2$). This A induces a language

$$L = \{0^n \mid 2^n = 2m^2 \text{ for some } m \text{ and } f(A^{=n}) = 1\}.$$

Let M_1, M_2, \dots be an enumeration of all oracle NQP-machines, with increasing polynomial time bounds (say, M_i has time bound $p_i(n) = n^i + i$). Such an enumeration exists because we can assume without loss of generality that the machines only use algebraic amplitudes [1, 5, 12]. At the start of our construction, A is the empty set. Going along $i = 1, 2, \dots$, for each M_i we will pick a specific input length n_i and define $A^{=n_i}$ in such a way that M_i^A will err on 0^{n_i} , and hence it will not accept L .

Consider M_i . Its running time is bounded by the polynomial $p_i(n)$ in the input length. Let n_i be the smallest input length such that (1) $2^{n_i} = 2m^2$ for some m , (2) $p_i(n_i) \leq m/2$, and (3) n_i is so large that for all $j < i$ we have $p_j(n_j) < n_i$.¹ Since M_i makes at most $p_i(n_i) < m/2 + 1 = NQ(f)$ queries to the bits of $x = A^{=n_i}$, Theorem 1 implies that M_i cannot be a nondeterministic algorithm for f . Hence there exists some $x \in \mathcal{X}_0 \cup \mathcal{X}_1$ where M_i errs: either $x \in \mathcal{X}_0$ while M_i has positive acceptance probability when $A^{=n} = x$; or $x \in \mathcal{X}_1$ while M_i has zero acceptance probability when $A^{=n_i} = x$. Define $A^{=n_i}$ to be that x . This ensures that M_i^A does not accept L .

Doing this for all M_i and filling the yet-undefined levels $A^{=n}$ by arbitrary promise-inputs to f , we now have a language L that is accepted by none of the M_i^A , hence $L \notin \text{NQP}^A$. On the other hand, the Deutsch-Jozsa algorithm implies $L \in \text{EQP}^{\text{ZPP}^A}$, so we have our separation. \square

5 Conclusion

We proved that the composition of two problems that are easy for zero-error quantum computing need not be easy itself. This contrasts strongly with the case of classical algorithms, and shows that our classical intuition about expected running time does not carry over very well to quantum algorithms. The problem in using a zero-error algorithm as a subroutine in a quantum algorithm seems to be that we cannot reverse the computation to obtain an answer without additional non-zero workspace. This remaining non-zero workspace then messes up later quantum interference in the main program. Being able to compose zero-error algorithms is a desirable property that obviously holds in the classical world. Unfortunately, this property does not hold in the quantum world.

References

- [1] L. M. Adleman, J. Demarrais, and M. A. Huang. Quantum computability. *SIAM Journal on Computing*, 26(5):1524–1540, 1997.
- [2] R. Beals, H. Buhrman, R. Cleve, M. Mosca, and R. de Wolf. Quantum lower bounds by polynomials. In *Proceedings of 39th IEEE FOCS*, pages 352–361, 1998. quant-ph/9802049.
- [3] H. Buhrman and R. de Wolf. Complexity measures and decision tree complexity: A survey. *Theoretical Computer Science*, 288:21–43, 2002.
- [4] D. Deutsch and R. Jozsa. Rapid solution of problems by quantum computation. In *Proceedings of the Royal Society of London*, volume A439, pages 553–558, 1992.

¹This third condition ensures that when we define $A^{=n_i}$ to thwart M_i , the behavior of earlier M_j s on input length n_j won't be changed (because M_j on input length n_j doesn't have enough time to query strings of length n_i).

- [5] S. Fenner, F. Green, S. Homer, and R. Pruim. Determining acceptance possibility for a quantum computation is hard for the polynomial hierarchy. In *Proceedings of the 6th Italian Conference on Theoretical Computer Science*, pages 241–252, 1998. quant-ph/9812056.
- [6] P. Høyer and R. de Wolf. Improved quantum communication complexity bounds for disjointness and equality. In *Proceedings of 19th Annual Symposium on Theoretical Aspects of Computer Science (STACS'2002)*, volume 2285 of *Lecture Notes in Computer Science*, pages 299–310. Springer, 2002. quant-ph/0109068.
- [7] A. Kitaev, A. Shen, and M. Vyalıy. *Classical and Quantum Computation*. American Mathematical Society, 2002.
- [8] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- [9] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.
- [10] R. de Wolf. Characterization of non-deterministic quantum query and quantum communication complexity. In *Proceedings of 15th IEEE Conference on Computational Complexity*, pages 271–278, 2000. cs.CC/0001014.
- [11] R. de Wolf. Nondeterministic quantum query and quantum communication complexity. *SIAM Journal on Computing*, 2002. Journal version of parts of [10] and [6]. To appear.
- [12] T. Yamakami and A. C-C. Yao. $NQP_C = co-C=P$. *Information Processing Letters*, 71(2):63–69, 1999. quant-ph/9812032.