Pure Nash Equilibria in Weighted Congestion Games

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### Congestion Model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$

- $N = \{1, \ldots, n\}$ finite set of players
- $F = \{1, \ldots, m\}$ finite set of facilities
- $X = \bigotimes_{i \in N} X_i$ set of strategy profiles with $X_i \subseteq 2^F$
- Set of cost functions $(c_f)_{f \in F}$ where $c_f : \mathbb{R}_{\geq 0} \to \mathbb{R}$
Weighted Congestion Game

Congestion model \( \mathcal{M} = (N, F, X, (c_f)_{f \in F}) \)

Vector of demands \( d = (d_i)_{i \in N}, \quad d_i \in \mathbb{R}_{>0} \)

**Weighted Congestion Game**

\[
G = (N, X, (\pi_i)_{i \in N})
\]

- private cost functions
  \[
  \pi_i(x) = \sum_{f \in x_i} d_i c_f(\ell_f(x))
  \]
- load
  \[
  \ell_f(x) = \sum_{i \in N : f \in x_i} d_i
  \]

Unweighted congestion game \( \Leftrightarrow d_i = 1 \) for all \( i \in N \)

Singleton Congestion Game \( \Leftrightarrow |x_i| = 1 \) for all \( i \in N, \quad x_i \in X_i \)
Definition: Pure Nash Equilibrium (PNE)

As strategy profile $x$ is a pure Nash equilibrium (PNE) if no player has an incentive to unilaterally change her decision:

$$
\pi_i(x_i, x_{-i}) \leq \pi_i(y_i, x_{-i}) \text{ for all } i \in N, x_i, y_i \in X_i, \text{ and } x_{-i} \in X_{-i}
$$
Unweighted Congestion Games ($d_i = 1$)

- PNE exists (via exact potential) [Rosenthal, IJGT ’73]
### Previous Work

#### Unweighted Congestion Games $(d_i = 1)$

- PNE exists (via exact potential) \cite{Rosenthal73}

#### Singleton Weighted Congestion Games $(|x_i| = 1)$

- non-decreasing cost functions, non-increasing cost functions
- PNE exists (via potential) \cite{Fotakis02, Even-Dar03, Fabrikant04, Rozenfeld06}
### Previous Work

**Unweighted Congestion Games** ($d_i = 1$)

- PNE exists (via exact potential) [Rosenthal, IJGT ’73]

**Singleton Weighted Congestion Games** ($|x_i| = 1$)

- non-decreasing cost functions, non-increasing cost functions
- PNE exists (via potential) [Fotakis et al., ICALP ’02] [Even-Dar et al., ICALP ’03] [Fabrikant et al., STOC ’04], [Rozenfeld & Tennenholtz, WINE ’06]

**Matroid Weighted Congestion Games**

- non-decreasing functions
  - PNE exists [Ackermann et al., WINE ’06]
## Affine Costs

- PNE exists (via exact potential) [Fotakis et al., ICALP '05]

## Exponential Costs

- PNE exist for $c_f(x) = \exp(x)$ for all $f \in F$
  [Spirakis and Panagopoulou, JEA '06]
- PNE exist for $c_f(x) = a_f \exp(\phi x) + b_f$ for all $f \in F$
  [H, Klimm, Möhring, SAGT '09]

## Counterexamples

- No PNE (2-players with demands $d_1 = 1, d_2 = 2$) [Libman & Orda, TS '01] [Fotakis et al., ICALP '05] [Goemans et al., FOCS '05]
Two Players with demands $d_1 = 1$, $d_2 = 2$

Two-wise linear costs

[Goemans et al., FOCS '05]

[Goemans et al., FOCS '05]
Two Players with demands $d_1 = 1$, $d_2 = 2$

Two-wise linear costs

Two-player Shapley cost sharing games ($c_e(x) = k_e / \ell_e(x)$) always have a PNE

[Anshelevich et al., SICOMP '08]
<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A set $\mathcal{C}$ of cost functions is <strong>consistent</strong> if every weighted congestion game to a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ with $c_f \in \mathcal{C}$ for all $f \in F$ admits a PNE.</td>
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Definition

A set $C$ of cost functions is **consistent** if every weighted congestion game to a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ with $c_f \in C$ for all $f \in F$ admits a PNE.

▷ Examples of consistent cost functions:

- $C = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(\ell) = a\ell + b, a, b \in \mathbb{R}\}$
- $C_\phi = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(\ell) = ae^{\phi \ell} + b, a, b \in \mathbb{R}\}$
Lemma (Monotonicity Lemma)

Let $C$ be a set of continuous cost functions. If $C$ is consistent, then $C$ contains only monotonic functions.

Even valid under the following restrictions

- games with 2 players
- games with 2 facilities
- singleton games
- games with identical cost functions on all facilities
- symmetric games
Proof of the Monotonicity Lemma

Consider game with $N = \{1, 2\}$, $F = \{f, g\}$, $d_1 = y - x$ and $d_2 = x$

- Player 1 prefers to be alone
- Player 2 prefers to share a facility with Player 1

$\Rightarrow$ no PNE

Game without PNE

<table>
<thead>
<tr>
<th></th>
<th>${f}$</th>
<th>${g}$</th>
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</thead>
<tbody>
<tr>
<td>${f}$</td>
<td>$(y-x)c(y)$</td>
<td>$xc(y)$</td>
</tr>
<tr>
<td>${g}$</td>
<td>$(y-x)c(y-x)$</td>
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Towards an Extended Monotonicity Lemma

**Model**

<table>
<thead>
<tr>
<th>2</th>
<th>{f}</th>
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<tbody>
<tr>
<td>1</td>
<td>{f}</td>
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## Towards an Extended Monotonicity Lemma

### Model

<table>
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<tr>
<th></th>
<th>${f, j}$</th>
<th>${h, g}$</th>
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<tbody>
<tr>
<td>${f, h}$</td>
<td><img src="#" alt="Orange Circle" /> $c_1$</td>
<td><img src="#" alt="Green Circle" /> $c_2$</td>
</tr>
<tr>
<td>${j, g}$</td>
<td><img src="#" alt="Green Circle" /> $c_2$</td>
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<tr>
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<td>$(y-x)(c_1(y) + c_2(y-x))$</td>
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<td>(F \cup H)</td>
<td>(c_1,</td>
<td>F</td>
</tr>
<tr>
<td>(J \cup G)</td>
<td>(c_2,</td>
<td>J</td>
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Towards an Extended Monotonicity Lemma

Consider cost function 
\[ c(\ell) = a_1c_1(\ell) - a_2c_2(\ell) \] 
and chose 
\[ d_1 = y - x, \quad d_2 = x \]

- Player 1 prefers to be alone
- Player 2 prefers to share

⇒ no PNE

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**Definition (Integer 2-hull)**

For a set $C$ of cost functions we call

$$L^2_N(C) = \{ c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \mid c(x) = a_1 c_1(x) - a_2 c_2(x), a_1, a_2 \in \mathbb{N}, c_1, c_2 \in C \}.$$ 

the integer 2-hull of $C$.

**Lemma (Extended Monotonicity Lemma 1)**

Let $C$ be a set of continuous cost functions. If $C$ is consistent then $L^2_N(C)$ contains only monotonic functions.

- Even valid for games with 2 players.
Generalizing Extended Monotonicity Lemma 1

Model

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</tr>
<tr>
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<td></td>
<td>( c_2,</td>
</tr>
<tr>
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<td></td>
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</tr>
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<td>J</td>
</tr>
<tr>
<td>G</td>
<td></td>
<td>( c_1,</td>
</tr>
</tbody>
</table>

- Introduce a third player with demand \( d_3 = b \) and a single strategy \( X_3 = \{ J \cup H \} \)
- Take \( c_1 = c_2 \)
- Effective cost on \( F \) and \( G \) equals \( c_1(x) \)
- Effective cost on \( H \) and \( J \) equals \( c_1(x + b) \)
Definition (Integer 3-hull)

For a set $C$ of cost functions we call

$$L_3^N(C) = \{ c : \mathbb{R}_{\geq 0} \to \mathbb{R} \mid c(x) = a_1 c_1(x) - a_2 c_1(x + b), \quad a_1, a_2, b \in \mathbb{N}, c_1 \in C \}.$$ 

the integer 3-hull of $C$.

Lemma (Extended Monotonicity Lemma 1)

Let $C$ be a set of continuous cost functions. If $C$ is consistent then $L_3^N(C)$ contains only monotonic functions.

▷ Even valid for games with 3 players.
Our Results so far

Singleton 2-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow$ $\mathcal{C}$ contains only monotonic functions

2-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow$ $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C}) = \{ c \mid c(x) = a_1 c_1(x) - a_2 c_2(x) \}$ contains only monotonic functions

3-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow$ $\mathcal{L}^3_{\mathbb{N}}(\mathcal{C}) = \{ c \mid c(x) = a_1 c_1(x + b) - a_2 c_1(x) \}$ contains only monotonic functions
Characterizing monotonicity of $\mathcal{L}_\mathbb{N}^2(C)$

$$\mathcal{L}_\mathbb{N}^2(C) = \{ c \mid c(x) = a_1 c_1(x) - a_2 c_2(x), \ a_1, a_2 \in \mathbb{N}, c_1, c_2 \in C \}$$

**Lemma**

Let $C$ be a set of twice continuously differentiable and monotonic functions. Then, $\mathcal{L}_\mathbb{N}^2(C)$ contains only monotonic increasing or decreasing functions iff for all $c_1, c_2 \in C$ there are $a, b \in \mathbb{R}$ such that $c_2(x) = ac_1(x) + b$ for all $x \geq 0$.

**Proof.**

" $\Leftarrow$ "  

$\tilde{c}(x) = a_1 c_1(x) - a_2 c_2(x)$  

$\tilde{c}(x) = a_1 c_1(x) - a a_2 c_1(x) - ab$  

$\tilde{c}'(x) = (a_1 - a a_2)c_1'(x)$

No change in sign of $\tilde{c}' \Rightarrow \tilde{c}$ is monotonic
Proof. (cont’d)

1. Show $D(x) := \det \begin{pmatrix} c'_1(x) & c'_2(x) \\ c''_1(x) & c''_2(x) \end{pmatrix} = 0$ for all $x \geq 0$.

- $D(x_0) \neq 0 \Rightarrow D(x) \neq 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$
- Non-trivial solution of

$$
\begin{pmatrix} c'_1(x) & c'_2(x) \\ c''_1(x) & c''_2(x) \end{pmatrix} \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{D(x)}{c''_2(x)} \end{pmatrix}.
$$

- $a_1(x) = 1$, $a_2(x)$ continuous $\Rightarrow$ find $x \in (x_0 - \epsilon, x_0 + \epsilon)$ with $p/q = a_2(x) \in \mathbb{Q}$
- $c = qc_1 - pc_2 \in L^2_{\mathbb{N}}(C)$ has strict local extremum in $x$
Proof. (cont’d).

2. Show $D(x) = 0$ for all $x \geq 0 \implies c_2(x) = ac_1 + b$
   
   ▶ If $c_1' \neq 0$ note that

   $$D(x) = 0 \implies \left(\frac{c_2'(x)}{c_1'(x)}\right)' = 0$$

   ▶ Integration delivers $c_2(x) = ac_1 + b$ for $a, b \in \mathbb{R}$
   
   ▶ Glueing together intervals with $c_1' = 0$ and $c_1' \neq 0$
   delivers the result
Let $C$ be a set of twice continuously differentiable functions. Then $C$ is consistent w.r.t. 2-player weighted congestion games iff the following holds:

1. $C$ contains only monotonic functions
2. for all $c_1, c_2 \in C$ there are constants $a, b \in \mathbb{R}$ such that $c_2 = ac_1 + b$

The if-part follows from a generalization of [H, Klimm, Möhring, SAGT '09].
Characterizing monotonicity of $\mathcal{L}^3_\mathbb{N}(C)$

$\mathcal{L}^3_\mathbb{N}(C) = \{ c \mid c(x) = a_1 c_1(x+b) - a_2 c_1(x), \ a_1, a_2, b \in \mathbb{N}, c_1 \in C \}$

[H, Klimm, ICALP '10]

**Theorem**

Let $C$ be a set of twice continuously differentiable functions. Then, $\mathcal{L}^3_\mathbb{N}(C)$ contains only monotonic functions iff one of the following holds

1. $C$ contains only affine functions
2. $C$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on $x$ while $\phi$ is independent of $c$.

The if-part follows from [Fotakis et al., ICALP '05], [H, Klimm, Möhring, SAGT '09], [Spirakis and Panagopoulou, JEA '06].
### Necessary conditions on consistency of costs

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<tr>
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<th>2-player</th>
<th>3-player</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singleton</td>
<td>monotonic</td>
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</tr>
<tr>
<td>Arbitrary</td>
<td>aff. transformations</td>
<td>affine or exponential</td>
</tr>
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If cost functions are strictly increasing and positive

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<td>[FIP $\leftrightarrow$ aff. or exp.]</td>
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<tr>
<td>Multi-commodity</td>
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**Red conditions are tight**