Approximation and Mechanism Design

Jason D. Hartline — Northwestern University

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Mechanism Design: how can a social planner / optimizer achieve objective when participant preferences are private.

Challenge: designer does not know participant preferences, participants may strategize when reporting preference!
Goals for Theory

Goals for Mechanism Design Theory:

- **Descriptive**: predict/affirm mechanisms arising in practice.
- **Prescriptive**: suggest how good mechanisms can be designed.
- **Conclusive**: pinpoint salient characteristics of good mechanisms.
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**Informal Thesis:** *approximately optimality* is often descriptive, prescriptive, and conclusive.
Example 1: Gambler’s Stopping Game

A Gambler’s *Stopping Game*:

- *sequence* of $n$ games,
- *prize* of game $i$ is distributed from $F_i$,
- *prior-knowledge* of distributions.

On day $i$, gambler plays game $i$:

- *realizes* prize $v_i \sim F_i$,
- chooses to keep prize and *stop*, or
- discard prize and *continue*. 
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**Question:** How should our gambler play?
Optimal Strategy:

- threshold $t_i$ for stopping with $i$th prize.
- solve with “backwards induction”.
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Discussion:

• *Complicated*: $n$ different, unrelated thresholds.
• *Inconclusive*: what are properties of good strategies?
• *Non-robust*: what if order changes? what if distribution changes?
• *Non-general*: what do we learn about variants of Stopping Game?
Threshold Strategies and Prophet Inequality

**Threshold Strategy**: “fix \( t \), gambler takes first prize \( v_i \geq t \)”. (clearly suboptimal, may not accept prize on last day!)
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**Theorem**: *(Prophet Inequality)* For $t$ such that $\Pr[\text{“no prize”}] = 1/2$,

\[ E[\text{prize for strategy } t] \geq E[\max_i v_i]/2. \]

[Samuel-Cahn ’84]
Threshold Strategies and Prophet Inequality

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Theorem: (Prophet Inequality) For $t$ such that $\Pr[\text{“no prize”}] = 1/2$,

$$E[\text{prize for strategy } t] \geq E[\max_i v_i] / 2.$$ [Samuel-Cahn ’84]

Discussion:

- **Simple**: one number $t$.
- **Conclusive**: trade-off “stopping early” with “never stopping”.
- **Robust**: change order? change distribution above or below $t$?
- **General**: same solution works for similar games: invariant of “tie-breaking rule”
0. Notation:

- \( q_i = \Pr[v_i < t] \).
- \( x = \Pr[\text{never stops}] = \prod_i q_i \).

1. Upper Bound on \( \mathbb{E}[\text{max}] \):

2. Lower Bound on \( \mathbb{E}[\text{prize}] \):

3. Choose \( x = 1/2 \) to prove theorem.
Prophet Inequality Proof

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E[\text{max}] \leq t + E[\max_i (v_i - t)^+] \]

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- Seller can always try ad hoc improvements on approximation.
   (e.g., single-item auctions)

2. Multi-dimensional Bayesian settings.
   (e.g., multi-item auctions)

3. Prior-free settings.
Part I: Approximation for single-dimensional Bayesian mechanism design

(where agent preferences are given by a private value for service, zero for no service; preferences are drawn from a distribution)
Example 2: Single-item auction

**Problem:** Bayesian Single-item Auction Problem

- a single item for sale,
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**Question:** What is optimal auction?
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7. **Cor:** for iid, regular dists, optimal auction is *Vickrey with monopoly reserve price* \( \varphi^{-1}(0) \).
Optimal Auctions:

- *iid, regular distributions*: Vickrey with monopoly reserve price.
- *general*: sell to bidder with highest positive virtual value.
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Discussion:

- iid, regular case: seems very special.
- general case: nobody runs optimal auction (too complicated?).
Approximation with reserve prices

Question: when is reserve pricing a good approximation?
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Thm: Vickrey with reserve = constant virtual price with 
\[ \Pr[\text{no sale}] = \frac{1}{2} \] is a 2-approximation. [Chawla, H, Malec, Sivan ’10]
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**Proof:** apply prophet inequality (tie-breaking by value) to virtual values.
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**Discussion:**

- constant virtual price \( \Rightarrow \) bidder-specific reserves.
- **simple:** reserve prices natural, practical, and easy to find.
- **robust:** posted pricing with arbitrary tie-breaking works fine, collusion fine, etc.
Anonymous Reserves

**Question:** for non-identical distributions, is *anonymous reserve* approximately optimal?

(e.g., eBay)
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**Thm:** non-identical, regular distributions, Vickrey with *anonymous reserve price* is 4-approximation. [H, Roughgarden ’09]
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**Proof:** more complicated extension of prophet inequalities.

**Discussion:**

- theorem is not tight, actual bound is in $[2, 4]$.
- justifies wide prevalence.
- approximation good for *platform design*. 
Beyond single-item auctions: *general feasibility constraints*. 
Extensions

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**Thm:** for non-identical regular distributions, VCG with monopoly reserves is often a 2-approximation.  
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Proof technique:

- optimal mechanism is a virtual surplus maximizer.

- reserve-price mechanisms are virtual surplus approximators.
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Proof technique:

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- reserve-price mechanisms are virtual surplus approximators.

**Basic Open Question:** to what extent do simple mechanisms approximate (well understood but complex) optimal ones?

**Challenges:** non-downward-closed settings, negative virtual values.
Part II: Approximation for multi-dimensional Bayesian mechanism design

(where agent preferences are given by values for each available service, zero for no service; preferences drawn from distribution)
Example 3: unit-demand pricing

**Problem:** Bayesian Unit-Demand Pricing

- a single, unit-demand consumer.
- $n$ items for sale.
- a dist. $F = F_1 \times \cdots \times F_n$ from which the consumer’s values for each item are drawn.

**Goal:** seller optimal *item-pricing* for $F$. 
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Question: What is optimal pricing?
Optimal Pricing: consider distribution, feasibility constraints, incentive constraints, and solve!
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Discussion:

• little conceptual insight and

• not generally tractable.
Analogy

Challenge: approximate optimal but we do not understand it?
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**Problem:** Bayesian Unit-demand Pricing (a.k.a., MD-PRICING)

- a single, *unit-demand* buyer,
- \( n \) items for sale, and
- a dist. \( F \) from which the consumer’s value for each item is drawn.

**Goal:** seller opt. item-pricing for \( F \).

**Problem:** Bayesian Single-item Auction (a.k.a., SD-AUCTION)

- a single item for sale,
- \( n \) buyers, and
- a dist. \( F \) from which the consumers’ values for the item are drawn.

**Goal:** seller opt. auction for \( F \).
**Analogy**

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**Goal:** seller opt. item-pricing for $F$.

**Goal:** seller opt. auction for $F$.

**Note:** Same informational structure.
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- conclusive: competition not important for approximation.
- practical: posted pricings widely prevalent. (e.g., eBay)
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Open Question: identify upper bounds beyond unit-demand settings that are

• conceptually tractable and
• approximable.
Part III: Approximation for prior-free mechanism design.

(mechanisms should be good for any set of agent preferences, not just given distributional assumptions)
The problem with priors

**Prior assumption:** the mechanism designer knows the distribution of agent preferences.
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Where does prior come from:

- historical data
  
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  (e.g. Coase Conjecture)

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**Question:** can we design good auctions without knowledge of prior-distribution?
Resource augmentation

Approach 1: “resource” augmentation.
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- “recruit one more bidder” is prior-free strategy.
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- **conclusive:** competition more important than optimization.
- **non-generic:** e.g., for \( k \)-unit auctions, need \( k \) additional bidders.
Special Case: $n = 1$

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![Diagram](image)
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![Diagram showing the relationship between Vickrey revenue and optimal reserve revenue]
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- So Vickrey with two bidders \( \geq \) optimal revenue from one bidder.
Example 4: digital goods

**Question:** how should a profit-maximizing seller sell a digital good (n bidder, n copies of item)?
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Bayesian Optimal Solution: if values are iid from known distribution, post the monopoly price $\varphi^{-1}(0)$. [Myerson ’81]
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Discussion:

- optimal,
- simple, but
- not prior-free
**Single-Sample Auction:** (for digital goods) [Dhangwatnotai, Roughgarden, Yan ’10]

1. pick random agent $i$ as sample.

2. offer all other agents price $v_i$.

3. reject $i$. 

Approximation via Single Sample
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**Discussion:**

- *prior-free.*
- *conclusive,* don’t need precise distribution, only need single sample for approximation. (more samples can improve approximation factor.)
- *generic,* applies to general settings.
Average-case vs Worst-case

**Note:** prior-free auction cannot be optimal in every setting.
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**Average Case Approximation:** \( \exists A, \forall F \in \text{IID}, \)

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E_{v \sim F}[A(v)] \geq \frac{E_{v \sim F}[\text{OPT}_F(v)]}{\beta}.
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**Notes:**

- worst-case approximation implies average-case approximation.
- \( \sup_{F \in \text{IID}} \text{OPT}_F(v) \) is *prior-free performance benchmark*.
- for digital goods, prior-free benchmark = optimal posted price revenue.
Random Sampling Auction: (for digital goods) [Goldberg, H, Wright ’01]

1. Randomly partition agents into two sets.
2. Compute optimal posted prices for each set.
3. Offer prices to opposite set.
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**Discussion:**

- **conclusive,** market analysis can be done “on the fly”
- **worst-case** is for $n = 2$.
- **practical,** bounds approach 1 in limit with $n$.
- **generic,** analysis extends beyond digital goods.
Extensions

Prior-free results extend to limited supply, downward-closed settings, non-identical distributions, other objectives, etc. [citations omitted]
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Open Questions:

- non-downward-closed settings?
- multi-dimensional settings?
- beyond the *revelation principle*?
Conclusions:

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**Basic Open Question:** attack economic impossibility w. approximation.