Reducing (Bayesian) Mechanism Design to Algorithm Design

Bobby Kleinberg
CWI Workshop on Advances in Algorithmic Game Theory
September 3, 2010

Joint work with: Jason Hartline, Azarakhsh Malekian
Petra’s Queue

1. Into Great Silence  High
2. Lulu & Jimi        Med.
3. Avatar             Low
4. The White Ribbon   High
5. Signs of Life      Low
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Petra’s Queue  

The White Ribbon High
Avatar Low
Speed Racer Low
Signs of Life High
Transformers 2 Low

Bobby’s Queue  

Avatar High
The Secret in Their Eyes Low
Sherlock Holmes High
Iron Man 2 High
Inception High
**Petra’s Queue**

- The White Ribbon: High
- Avatar: Low
- Speed Racer: Low
- Signs of Life: High
- Transformers 2: Low

**Bobby’s Queue**

- Avatar: High
- The Secret in Their Eyes: Low
- Sherlock Holmes: High
- Iron Man 2: High
- Inception: High

Tuesday, September 7, 2010
Our Guiding Question

*Is there a reduction that makes an arbitrary algorithm incentive compatible, with little or no loss in social welfare?*
Preliminaries

$X_i = \text{type space}$ \quad $v_i : X_i \times Y \to \mathbb{R}$ (valuation)

$X = X_1 \times \cdots \times X_n$ \quad $f : X \to Y$ (allocation)

$Y = \text{outcomes}$ \quad $p_i : X \to \mathbb{R}$ (payment)

Objective: $\max_y \{ \sum v_i(x_i, y) \}$ “social welfare”

Truthfulness: Two different notions...
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Truthfulness: Two different notions...

Dominant Strategy \( \forall i \ \forall x_{-i} \ v_i(x_i,f(\cdot,x_{-i}))-p_i(\cdot) \max@x_i \)

Bayesian \( \forall i \ E[v_i(x_i,f(\cdot,x_{-i}))]-p_i(\cdot) \max@x_i \)
## Cyclic Monotonicity

<table>
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<th>$x_1$</th>
<th>$y_1 = f(x_1)$</th>
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<td>$x_2$</td>
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Truthfulness of single-player mechanism \((f,p)\) implies “max matching property” of \(f\).

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**Mechanism design is algorithm design with a cyclic monotonicity constraint.**
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Is there a reduction that makes an arbitrary algorithm incentive compatible, with little or no loss in social welfare?
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Is there a reduction that makes an arbitrary algorithm incentive compatible, with little or no loss in social welfare?

Yes, VCG.
Our Guiding Question

Is there an efficient reduction that makes an arbitrary algorithm incentive compatible, with little or no loss in social welfare?
Our Guiding Question

*Is there an *efficient* reduction that makes an arbitrary algorithm incentive compatible, with little or no loss in social welfare?*

**Dominant Strategy**  
No  
[work of Dobzinski, Lavi, Mu’alem, Nisan, Papadimitriou, Schapira, Singer, many others]

**Bayesian**  
Yes!  
[Hartline-Lucier STOC’10, this talk]
Assumptions

How are players’ bid distributions specified?

1. General oracles for sampling $x_i$, evaluating $v_i$
2. Single-parameter $X_i \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}^n$, $v_i(x_i,y) = x_i \cdot y_i$
3. Discrete sample space $\Omega$ \quad input size = $|\Omega|$

How is the algorithm $f$ specified?

1. Black box model \quad oracle for evaluating $f$
2. Ideal model \quad additional oracle for $\mathbb{E}[f(x,x_i)]$
### Summary of Results

Assume $v_i : X_i \times Y \to [0,1]$, and seek $\varepsilon$-additive approximation to social welfare of $f$.

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Table gives sample complexity $s$.

Running time is $O(ns^3)$.
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* $\Delta$ = “number of parameters to specify a type or outcome”
** Mechanism is only $\varepsilon$-truthful, not truthful.
The Fine Print

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<td>$\tilde{\mathcal{O}}(\varepsilon^{-3})^{**}$</td>
<td>$\tilde{\mathcal{O}}(\varepsilon^{-3\Delta-6})^{**}$</td>
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∀ sufficiently large $k$, each $X_i$ can be covered by $O(k^\Delta)$ sets of diameter $1/k$ in the $L_\infty$ metric. (Distance between types is max. difference of values they assign to the same outcome.)

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Idea #1: Surrogates

- Replace each bid $x_i$ with a random surrogate $\sigma(x_i)$.
- Choose outcome $y = f(\sigma(x_1),...,\sigma(x_n))$.

We require two properties of the sampling process $\sigma(\cdot)$.

**Stationarity:** stationary distrib. is the type distrib. of player i.

**Monotonicity:** the function $x \rightarrow \sigma(x)$ is CMON.
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Allocation rule $RS(\sigma)$

\[
\nu(x,y) = \mathbb{E}[v(x, f(y, x_{-i}))]
\]

The expected value that type $x$ assigns to the random outcome obtained using surrogate $y$. 

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Idea #1: Surrogates

Stationarity: stationary distrib. is type distrib. of player i.

Monotonicity: the function $x \rightarrow \sigma(x)$ is CMON.

Theorem: If $\sigma$ satisfies these two properties, then the allocation rule $RS(\sigma)$ is CMON.

Proof:
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Remark: Easy to compute payments for $RS(\sigma)$, but won’t discuss the issue further in this talk.

Examples

1. $\sigma = \text{Id}$ satisfies stationary, but not monotonicity unless $f$ is monotone.

2. $\sigma = \text{Resample}$ satisfies both properties, but has lousy social welfare.
Idea #2: Replicas

1. Sample *replicas* $r_1, ..., r_m$ and *surrogates* $s_1, ..., s_m$ i.i.d. from type distribution on $X_i$.

2. Choose random $k$, set $r_k = x_i$.

3. Set edge weights $w_{ij} = \nu(r_i, s_j)$.

4. Let $\mu = \text{max-weight matching}$.

5. Declare surrogate $\sigma(x_i) = \mu(r_k)$. 

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Stationarity: Distrib. of $\mu(r_k)$ unchanged if step 2 omitted.
Idea #2: Replicas

1. Sample *replicas* \( r_1, \ldots, r_m \) and *surrogates* \( s_1, \ldots, s_m \) i.i.d. from type distribution on \( X_i \).

2. Choose random \( k \), set \( r_k = x_i \).

3. Set edge weights \( w_{ij} = \nu(r_i, s_j) \).

4. Let \( \mu = \text{max-weight matching} \).

5. Declare surrogate \( \sigma(x_i) = \mu(r_k) \).

Monotonicity: Conditional on replicas, surrogates, and \( k \), the mapping from \( x_i \) to \( \sigma(x_i) \) is monotone. (in fact, \( \mu \)'l in range)
Welfare Approximation

Welfare loss of bidder i is
\[ \nu(r_k, r_k) - \nu(r_k, \mu(r_k)) \]

Expectation is
\[ \frac{1}{m} \left[ \sum_k \nu(r_k, r_k) - \sum_k \nu(r_k, \mu(r_k)) \right] \]

This is no greater than
\[ \frac{1}{m} \left[ \sum_k \nu(r_k, r_k) - \sum_k \nu(r_k, \lambda(r_k)) \right] \]
for any other matching \( \lambda \).

Bound this from above by
\[ \frac{1}{m} \left[ \sum_k ||r_k - \lambda(r_k)||_\infty \right] \]

Choose \( \lambda \) to minimize the RHS.
Transportation Cost

X a metric space.

Transportation cost between two m-point subsets of X is length of min-cost matching.
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$X$ a metric space.

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Transportation cost between two m-point subsets of X is length of min-cost matching.

Theorem: If Diam(X)=1 and X partitions into $k^\Delta$ sets of diameter $1/k$, the expected transportation cost of two random m-point subsets is $O(m/k + (mk^\Delta)^{1/2})$.

Proof Sketch: Match as many points as possible to partners in same piece of partition. Bound expected number of unmatched points by $(mk^\Delta)^{1/2}$. 

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**Corollary:** Replica-surrogate matching mechanism achieves $O(\varepsilon)$ welfare loss when $k = \varepsilon^{-1}$, $m = \varepsilon^{-\Delta-2}$.

**Remark:** More careful analysis gives $m = \varepsilon^{-\Delta-1}$ in doubling dimension $\Delta$. This is tight except for $\Delta \leq 2$. 
Extensions

- Improved mechanism for single-parameter case. \(\{\text{Replicas}\} = \{\text{Surrogates}\}\)
- Mechanisms for the black box model. (Can evaluate \(f\) but can’t query its exact expectation.)
- Single-parameter case
- Discrete type space
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### Open Questions

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* $\Delta$=covering dimension
** $\varepsilon$-truthful, but not truthful

- Exponential dependence on $\Delta$ necessary?
- Remove the double-asterisk ... please!!
- Achieve $\varepsilon$-approximation pointwise, not in expectation.
- Approximate other objectives, e.g. fairness.