Combinatorial Auctions with Budgets

Joint work with Amos Fiat, Jared Saia and Piotr Sankowski

Stefano Leonardi

Sapienza University of Rome

Advances in Algorithmic Game Theory

CWI Amsterdam - September 2 - 3, 2010

Outline

- Introduction
- Multi-unit Auctions with Budgets
- Combinatorial Auctions with Budgets
- Pareto Optimality
- Conclusions

Auctions with Budgets

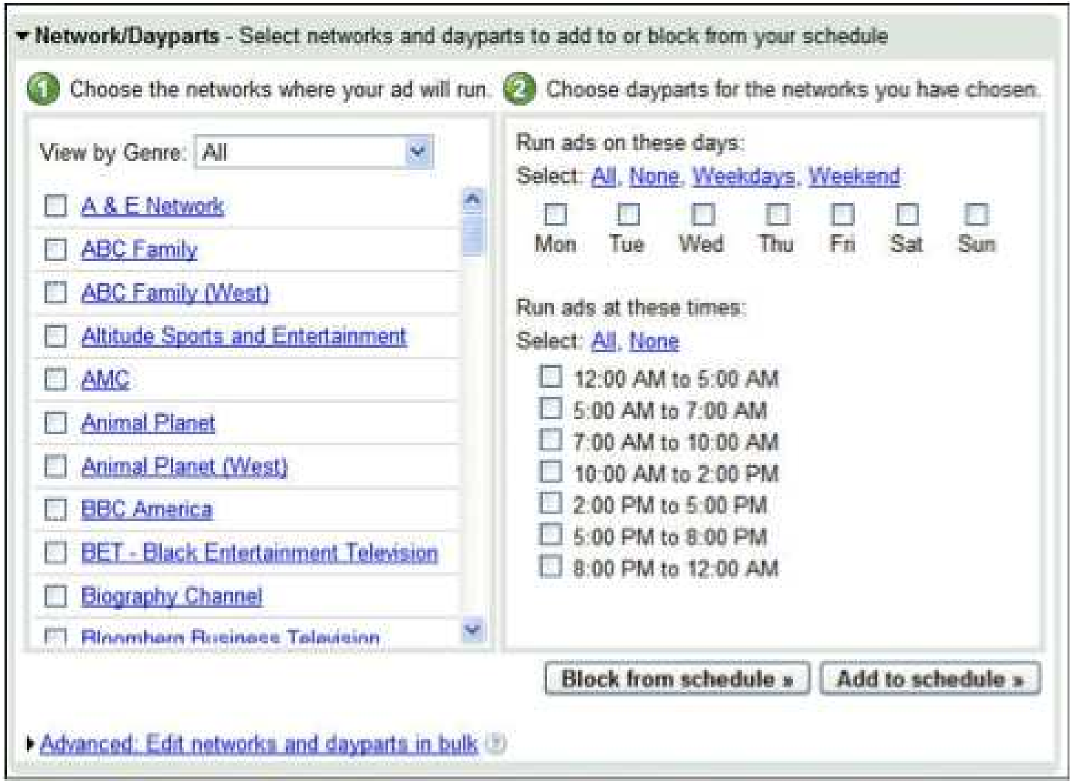
Auctions with Budgets

- Auctions are run daily from Google and other companies of on-line advertising
- Google sells TV ads through a Web interface

Advertisers specify the following parameters:

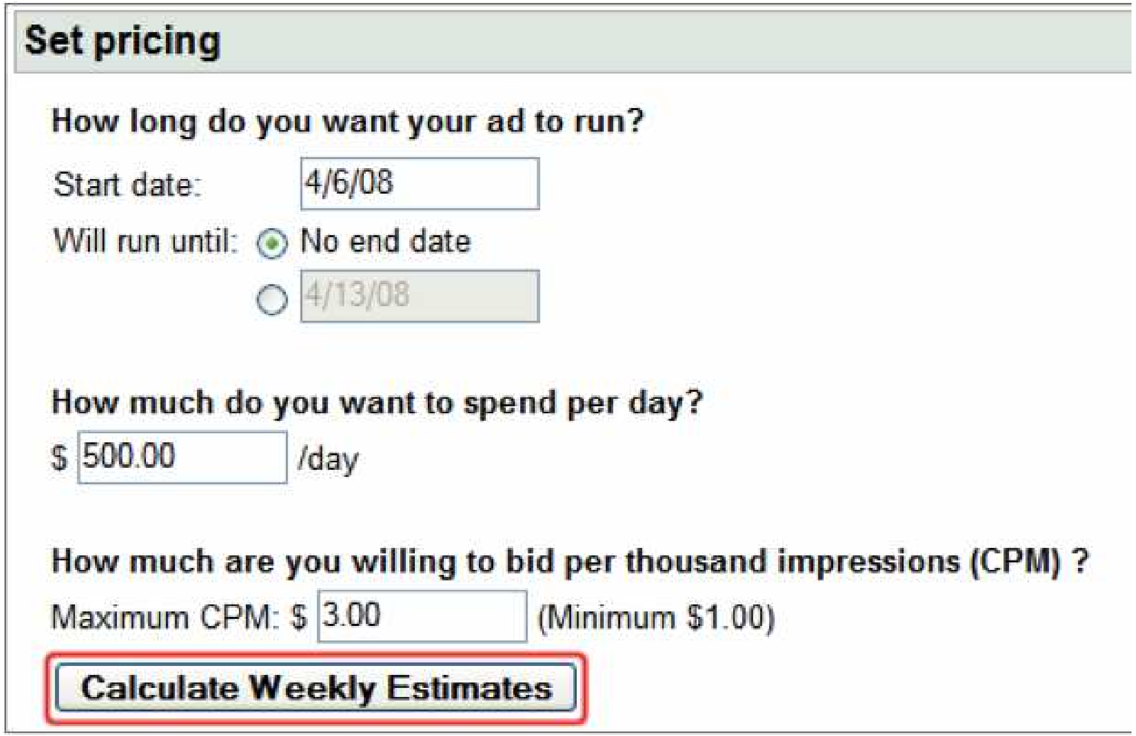
- Target TV shows
- Daily budget limit
- Valuation per impression

Google TV Ads



From Noam Nisan’s ICALP talk on Google TV Ads

Google TV Ads



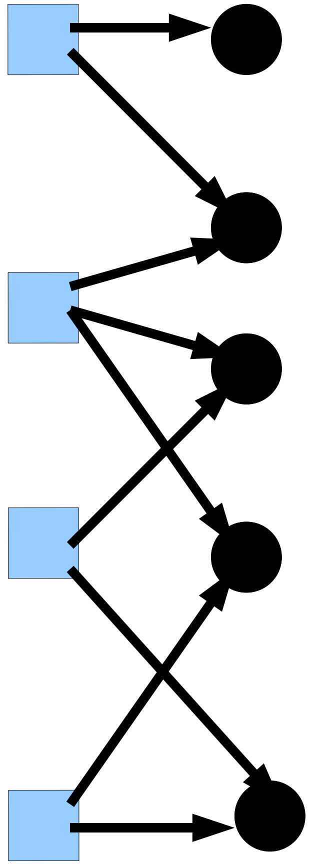
From Noam Nisan’s ICALP talk on Google TV Ads

Combinatorial Auctions with Budgets

The model:

- There is a set A of n agents (advertisers) and m items (slots)
- Agent i is interested in a subset S_i of the items
- Agent i has budget b_i and valuation $v_i > 0$ for each item in S_i

Valuations, budgets and sets S_i are private knowledge of the agents.



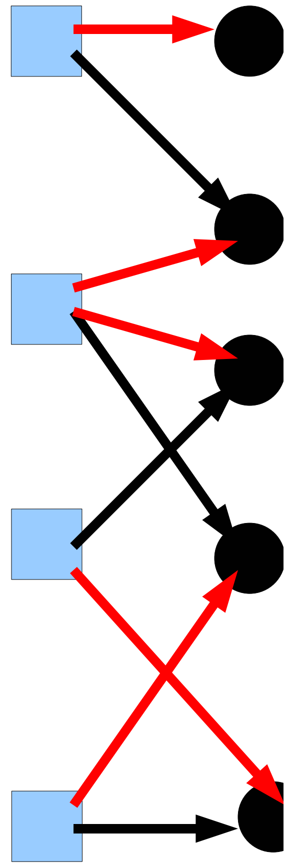
Combinatorial Auctions with Budgets

The Auctioneer:

- Assign $M(i)$ items from S_i to agent i and payment $P(i)$
- Utility for agent i (Additive - Non quasi-linear):

$$\begin{cases} M(i)v_i - P(i) & \text{if} P(i) \leq b_i \\ -\infty & \text{if} P(i) > b_i \end{cases}$$

- The utility for the auctioneer is $\sum_{j=1}^{n} P(j)$



No Budgets: Vickrey Auction

- Assume identical items and agents with infinite budget
- Vickrey auction allocates item to agent with highest valuation for item
- Item price is second highest valuation

Properties of Vickrey:

Maximize

social welfare	=	total valuation of the agents
	=	total utility of the agents and of the auctioneer

Truthfulness: bidding real valuation is a dominant strategy

Auctions with Budgets

Example: 2 agents, 50 identical units:

- Alice has valuation \$20 and budget \$50
- Bob has valuation \$5 and budget \$150
- Vickrey would sell all 50 items to Alice at price of \$ 250
- Auctions with budgets are not quasi-linear. Therefore maximizing sum of utilities does not correspond to maximizing sum of the valuations
- Indeed, there are no truthful auctions with budgets that maximize social welfare

Maximizing social welfare is not attainable!

A weaker objective is Pareto optimality:

There exist no allocation with all agents better off (including the Auctioneer)

Multi-unit Auctions with Budgets

Multi-Unit Auctions: for all i, j, $S_i = S_j$

- There are no truthful auctions that are Pareto optimal for multi-unit auctions with budgets [Dobzinski, Lavi, Nisan, FOCS 2008]
- There exists an ascending auction [Ausubel, American Economic Review 2004] that is truthful if budgets are public knowledge [DLN08]
- The ascending auction is Pareto-optimal [DLN08]!
- Lots of follow-up research in the last 2 years

A major open problem posed in [DLN08] was to derive a similar result for combinatorial auctions

There exists a Pareto-optimal truthful combinatorial auction for single-valued agents with private valuations [Fiat, L., Sankowski, Saia, 2010]

The Multi-unit Auction with Budgets

The Multi-unit Auction with Budgets

Denote by m the current number of items.

Demand of i at price p: $D_i(p) = \begin{cases} \texttt{min}\{m, \lfloor b_i/p \rfloor\} & \text{if} p \leq v_i \\ 0 & \text{if} p > v_i \end{cases}$

Demand of i at price p^+: $D_i^+(p) = \lim_{\epsilon \to 0^+} D_i(p+\epsilon)$

As price goes up demands go down because

1. Budget is limited, Or
2. Price hits valuation and demand drops to 0

The auction sells an item to some agent a at price p if

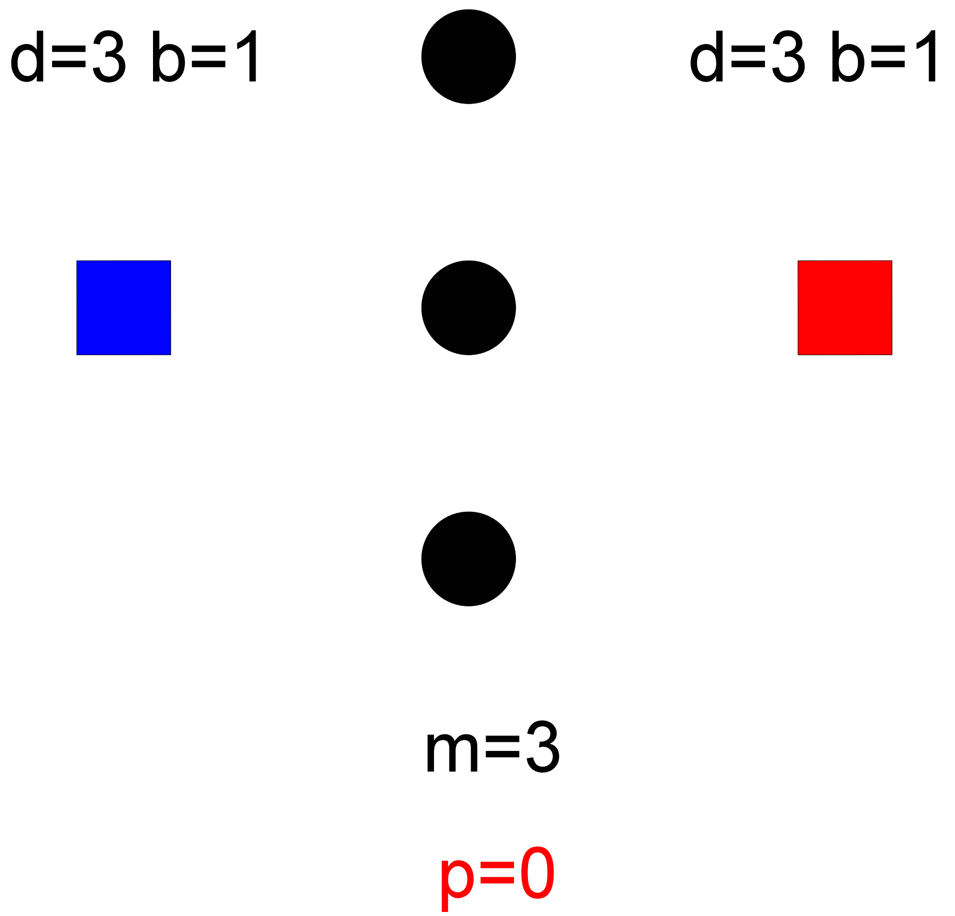
- (Truthfulness): excluding a, all other agents cannot purchase all items at price p or higher: $\sum_{i \in A/a} D_i(p) < m$, Or,
- (Sell all items): at any higher price some items will never be sold: $\sum_{i \in A} D_i^+(p) < m$

Multi-unit Auction with Budgets

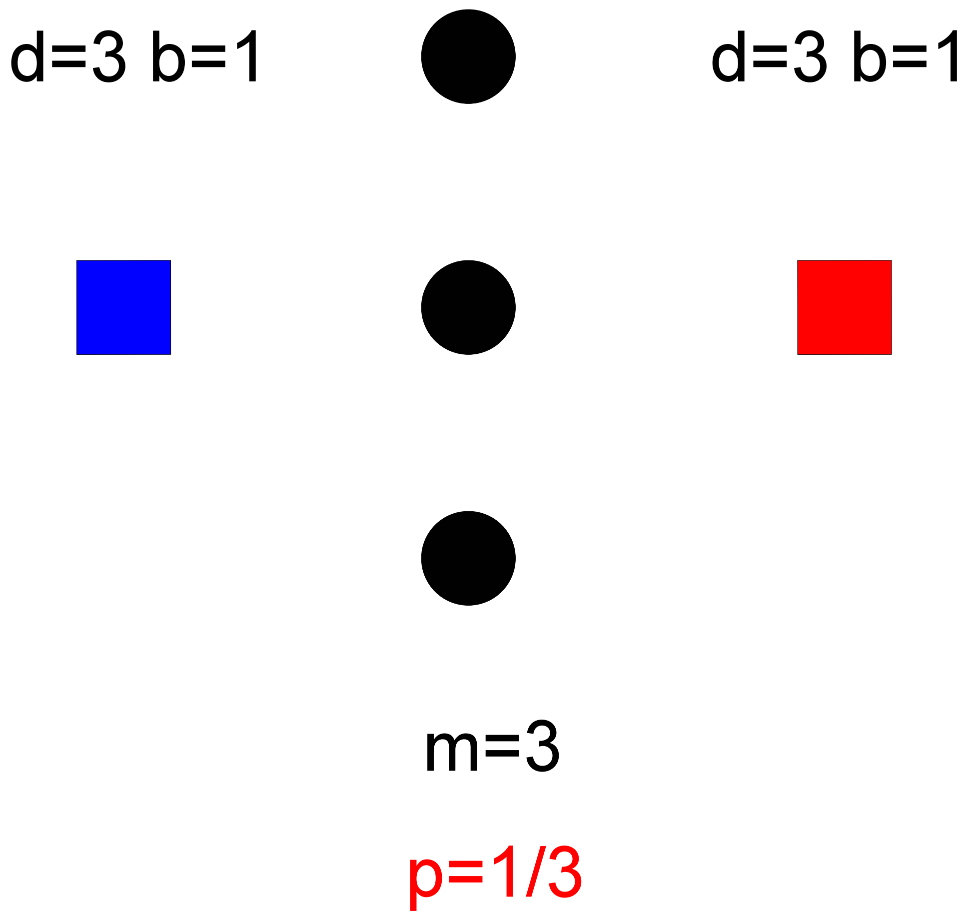
Valuation limited agents: $V = \{i : D_i(p) > 0 \text{ and } p = v_i\}$

```
procedure MULTI-UNIT AUCTION WITH BUDGETS(v, b)
    p ← 0, ∀i, d_i = D_i(0)
    while (A ≠ ∅) do
        Sell(V)
        A=A-V
        repeat
            if ∃i : d(A/i) < m then Sell(i)
            else
                For arbitrarily agent i with d_i > D_i^+(p) : d_i ← D_i^+(p)
            end if
        until ∀i: (d_i = D_i^+(p)) and (d(A/i) ≥ m)
        Increase p until for some i, D_i(p) ≠ D_i^+(p)
    end while
end procedure
```

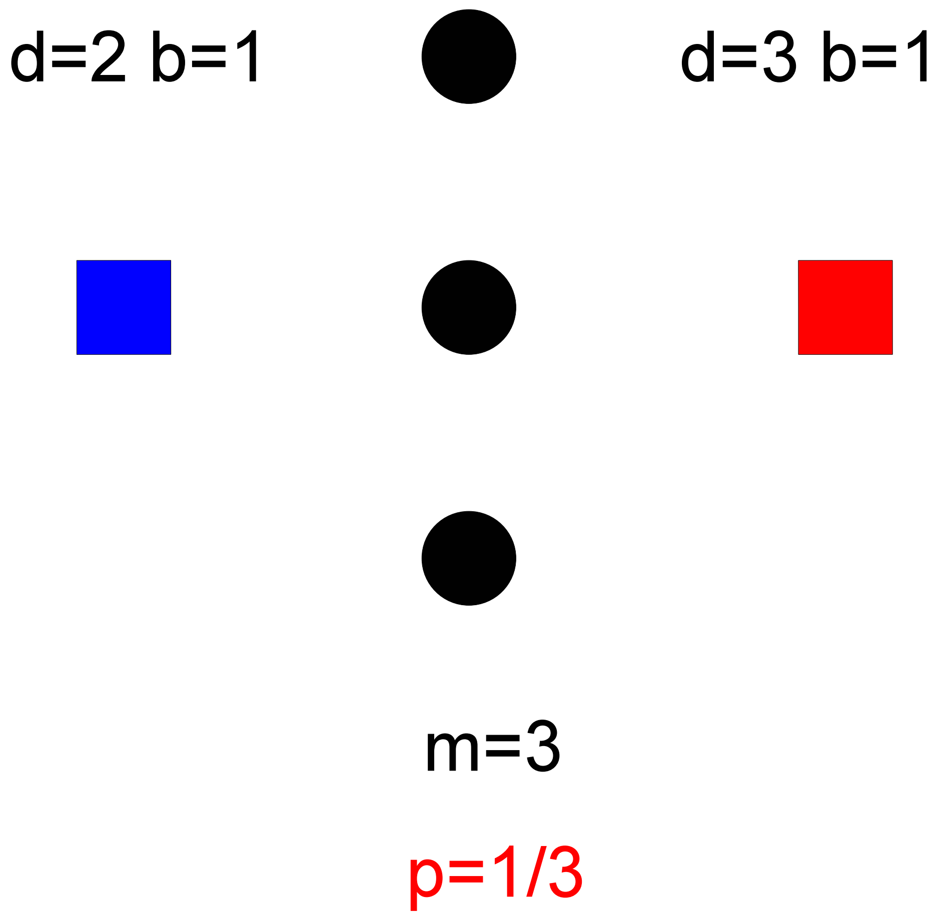
Example of Ascending Auction



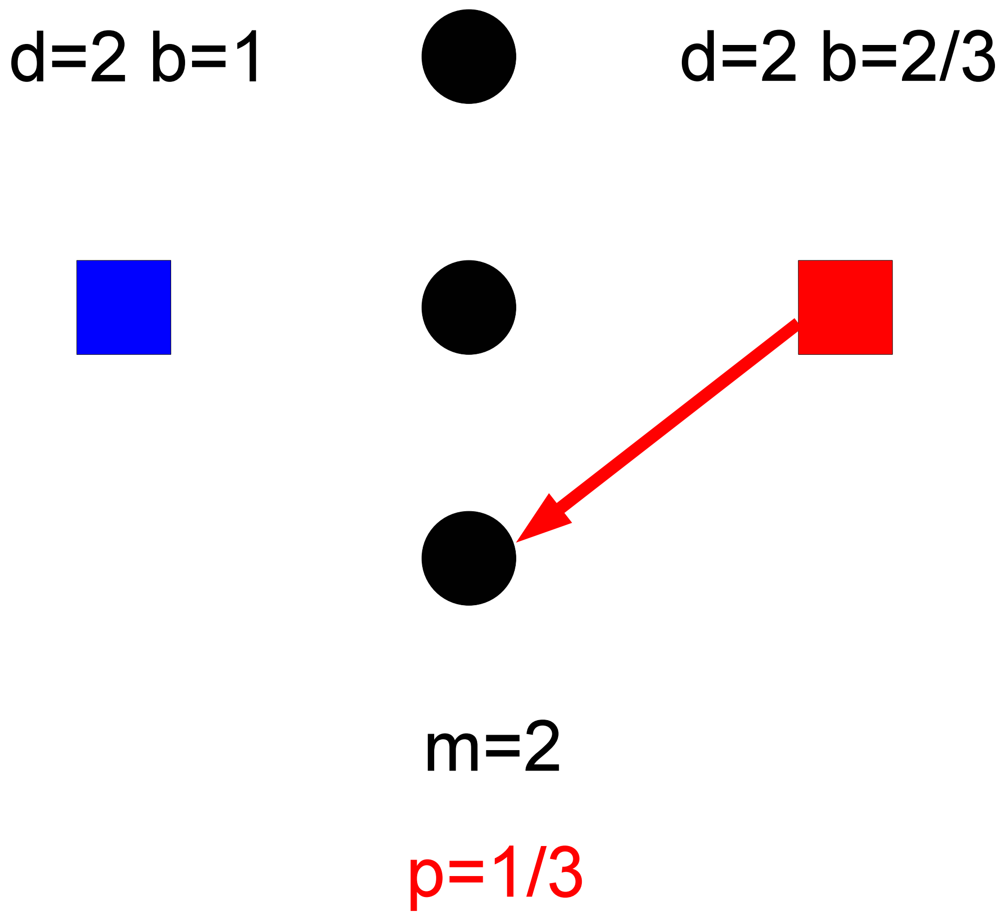
Example of Ascending Auction



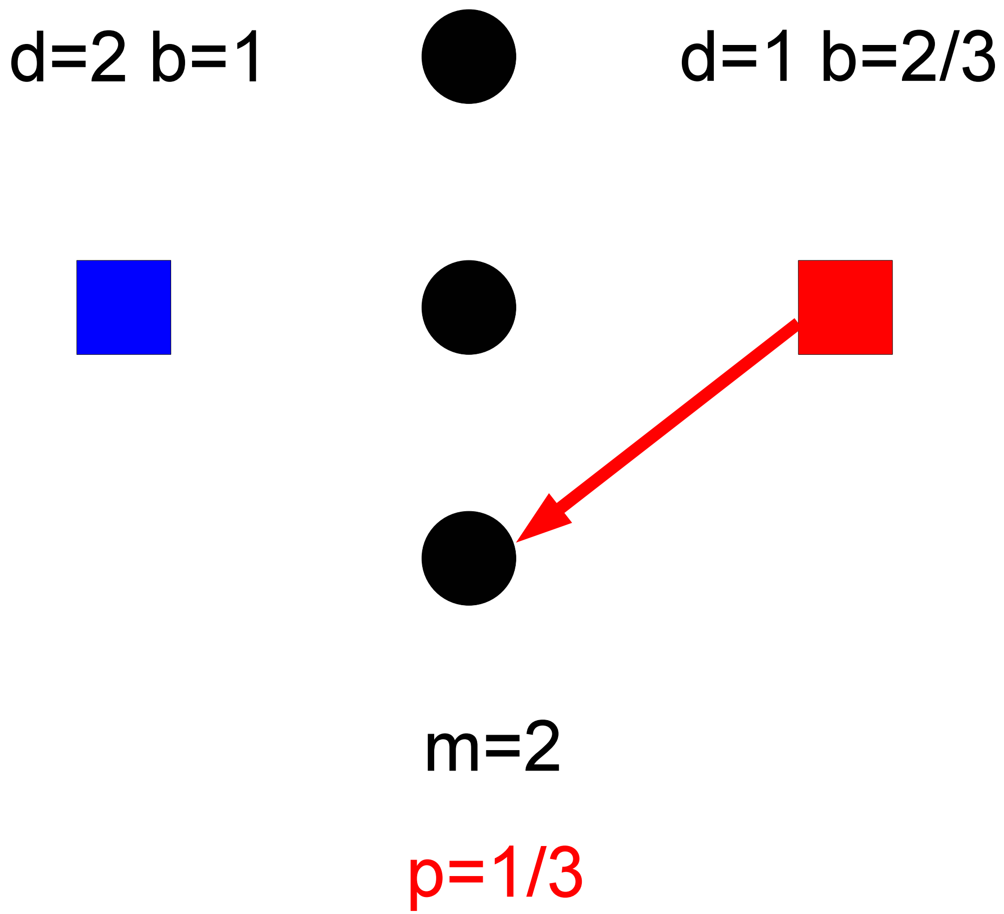
Example of Ascending Auction



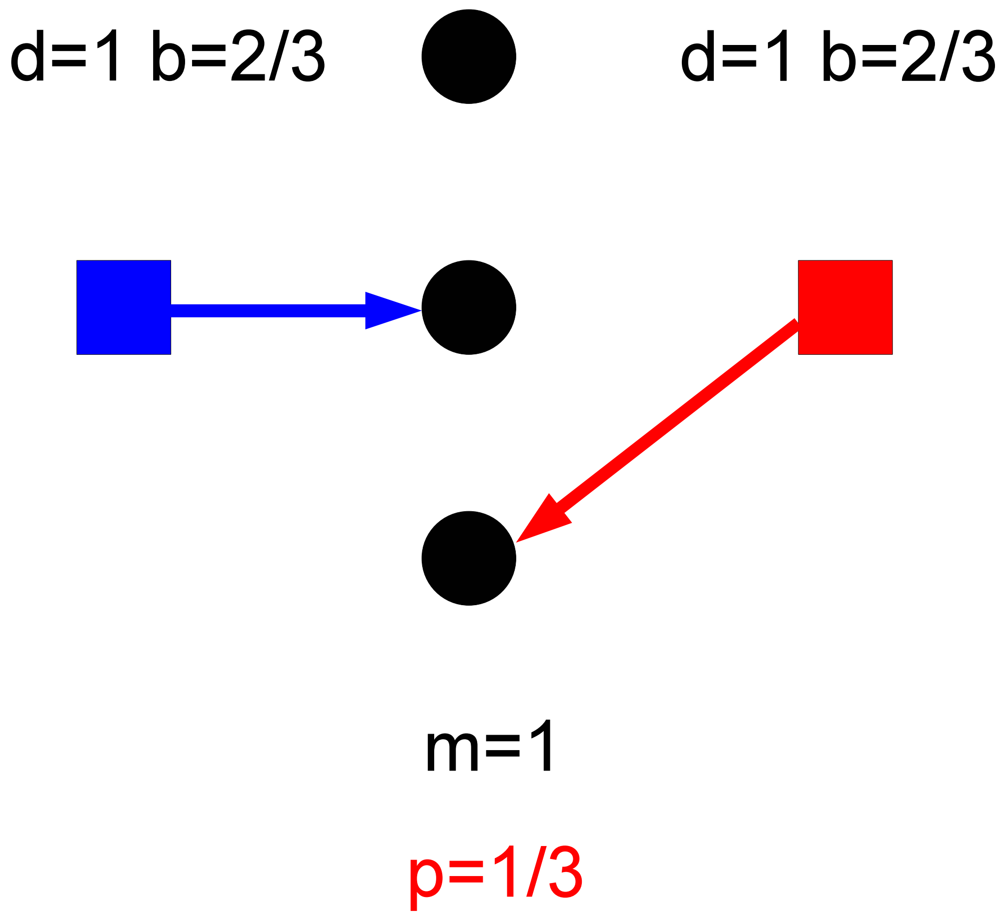
Example of Ascending Auction



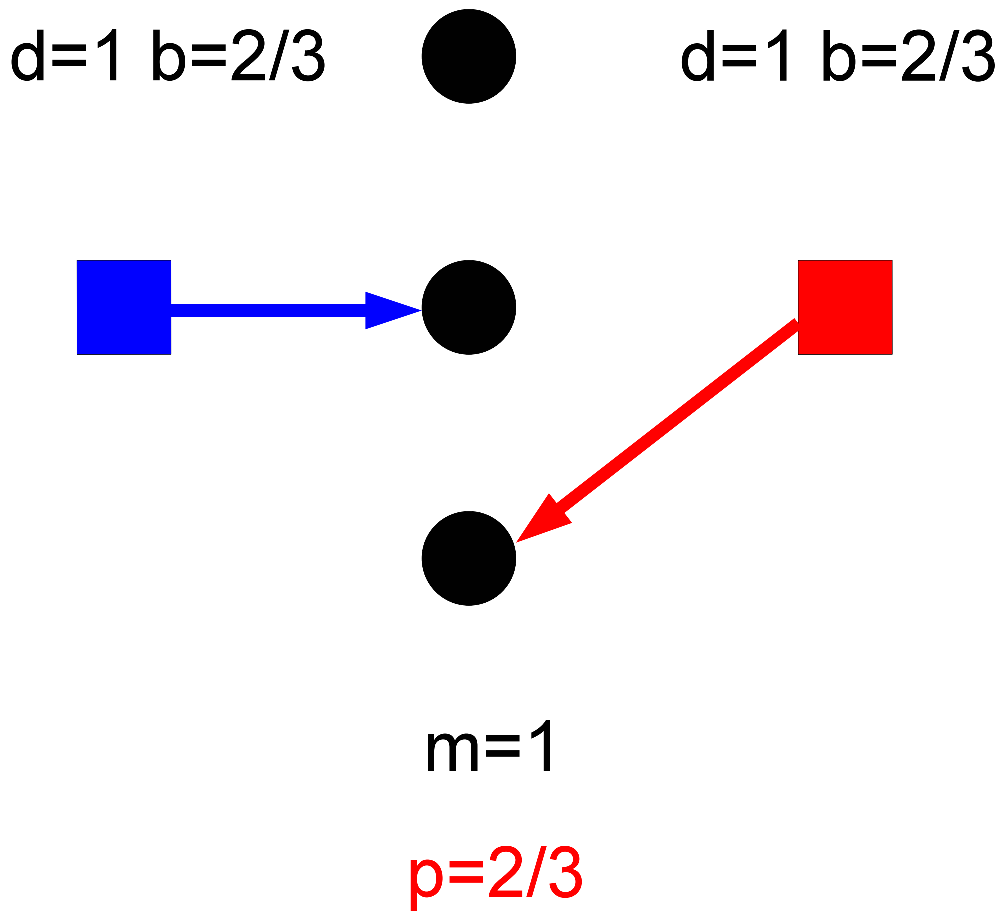
Example of Ascending Auction



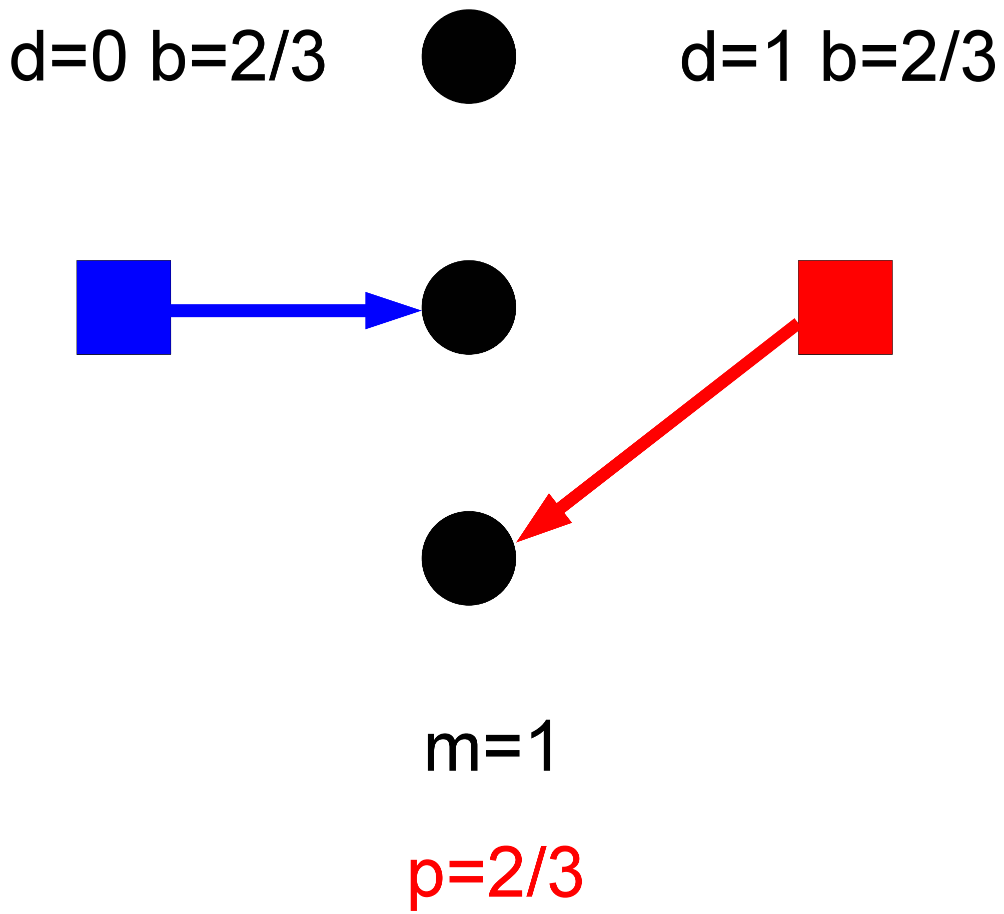
Example of Ascending Auction



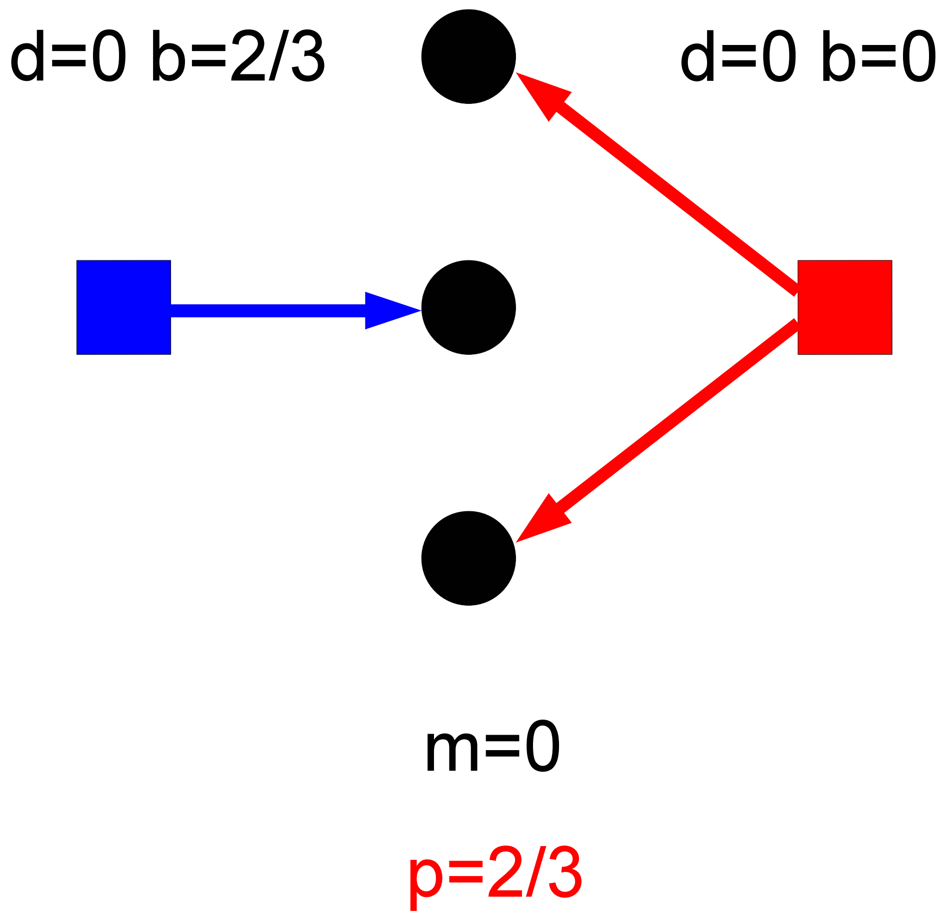
Example of Ascending Auction



Example of Ascending Auction



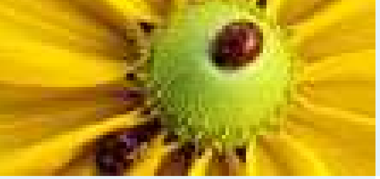
Example of Ascending Auction



Conditions for Pareto Optimality

Necessary condition for Pareto Optimality: All items are sold....

1. Special handling of Value-Limited Agents: if $d(V - A) < m$ then first sell to agents of A.
2. If we set all $d_i = D_i^+(p)$ it may result in $d(A) < m$: decrease demand agent by agent in arbitrary order so that m decreases only by 1 unit at a time.



Conditions for Pareto Optimality

Sufficient condition for Pareto Optimality (*no trade property*) for Multi-unit Auction.

There exist no two agents i, j, i allocated with at least 1 item, such that:

- $v_j > v_i$
- remaining budget of j: $b_j \geq v_i$

The ascending multi-unit auction is Pareto optimal [DLN08]

Show that the sufficient condition holds.

Proof of Pareto Optimality for Multi-unit Auction

Proof by contradiction.

Assume there exists two bidders i, j, such that $v_j > v_i$ and $b_j \geq v_i$.

Consider the last item sold to agent i. Agent $j \notin V$ at this time. Define M_k to be the number of items allocated to agent k at later time. There are two cases:

1. Agent $i \in V$ when it receives the item. Before Sell(V),

$$
\begin{aligned}
m &= \text{\# items to be sold to } V + \sum_{k \in A/\{V \cup j\}} M_k + M_j \\
&< \text{\# items to be sold to } V + \sum_{k \in A/\{V \cup j\}} D_k + D_j
\end{aligned}
$$

$$D_j \geq M_j + 1, \forall k, D_k \geq M_k.$$

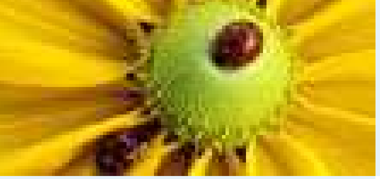
Proof of Pareto Optimality for Multi-unit Auction

2. Agent $i \notin V$ when it receives the item. Before Sell($i|d(A/i) < m$),

$$
\begin{aligned}
m &= \text{\# items to be sold to } i + \sum_{k \in A/\{i \cup j\}} M_k + M_j \\
&< \text{\# items to be sold to } i + \sum_{k \in A/\{i \cup j\}} d_k + d_j
\end{aligned}
$$

$$d_j \geq M_j + 1, \forall k, d_k \geq M_k.$$

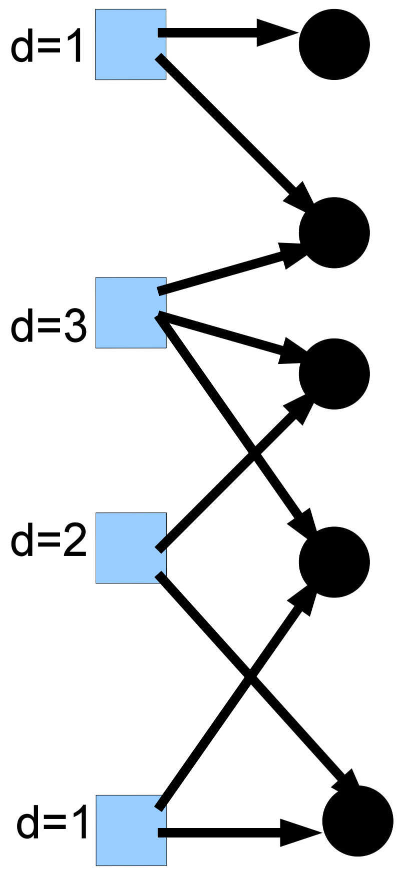
The Combinatorial Auction with Budgets



The Demand Graph

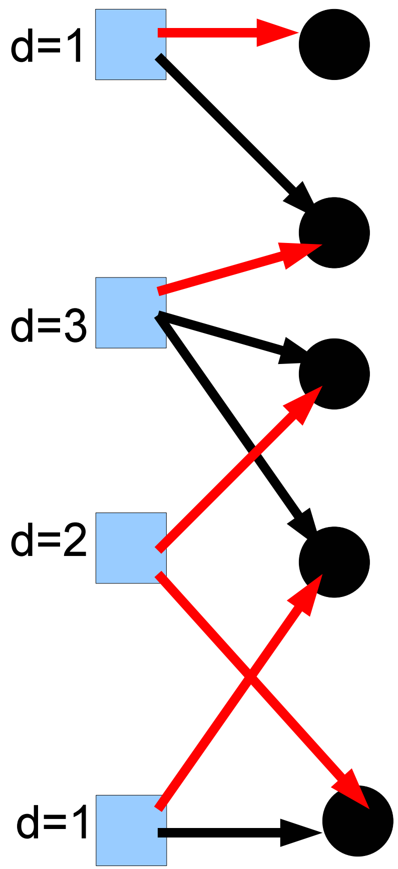
Demand graph: a bipartite graph G with all agents on the left, all items on the right, and edges (i, j) iff $j \in S_i$.

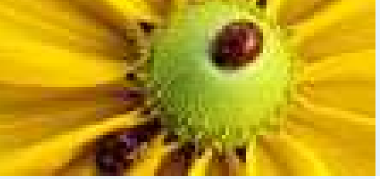
d-capacitated demand graph: every agent i has associated capacity d_i, every unsold item has capacity 1.



Matchings

Full matching in a d-capacitated demand graph: Matching of [possibly multiple] items to agents such that all items are matched and capacities are observed





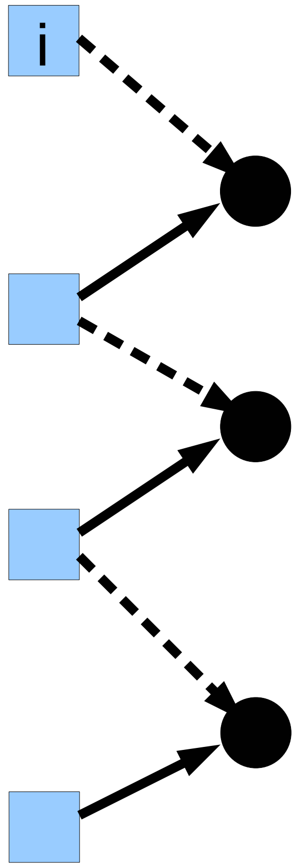
S-Avoid Matchings and Selling items

For a subset of agents S, a full S-avoid matching in a d-capacitated demand graph assigns a minimal number of items to agents in S.

A full S-Avoid matching in a d-capacitated demand graph can be computed using min-cost max-flow.

Let $B(\neg S)$ be the number of items assigned to agents not in S in a full S-Avoid matching

Sell(S) computes such an S-Avoid matching and for every (i, j) in this matching, $i \in S$, sells item j to agent i at current price.



i-AvoidMatching

Recap:

Demand of i at price p: $D_i(p) = \begin{cases} \mathtt{min}\{m, \lfloor b_i/p \rfloor\} & \text{if} p \leq v_i \\ 0 & \text{if} p > v_i \end{cases}$

Demand of i at price p^+: $D_i^+(p) = \lim_{\epsilon \longrightarrow 0^+} D_i(p+\epsilon)$

The Combinatorial Auction with Budgets

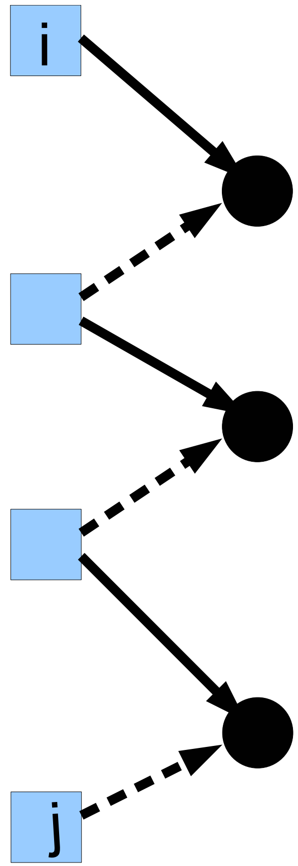

```
procedure COMBINATORIAL AUCTION WITH BUDGETS(v, b, {S_i})
    p ← 0
    while (A ≠ ∅) do
        Sell(V)
        A=A-V
        repeat
            if ∃i|B(¬{i}) < m then Sell(i)
            else
                For arbitrarily agent i with d_i > D_i^+(p) : d_i ← D_i^+(p)
            end if
        until ∀i: (d_i = D^+(i)) and B(¬{i}) ≥ m)
        Increase p until for some i, D_i(p) ≠ D_i^+(p))
    end while
end procedure
```

Trading Paths

Given an allocation (M, P), an **alternating path** for matching M: an even length path in the demand graph with all odd edges in M.

A **trading path** in allocation (M, P) is an alternating path from agent i to agent j such that:

- $v_j > v_i$
- remaining budget of j: $b_j \geq v_i$



No trading paths $\Leftrightarrow$ Pareto-Optimality

Theorem 1 *An allocation (M, P) is Pareto-optimal if and only if*
1. *All items are sold in (M, P), and*
2. *There are no trading paths in G with respect to (M, P).*

Proof: (only if) — Assume there exists a trading path in the demand graph G with respect to (M, P):

$$\pi = (a_1, t_1, a_2, t_2, \ldots, a_{j-1}, t_{j-1}, a_j)$$

as $v_{a_j} > v_{a_1}$ and $b^*_{a_j} \geq v_{a_1}$ then
- decrease payment of a_1 by v_{a_1}
- increase payment of a_j by v_{a_1}, and
- move item t_i from a_i to a_{i+1} for $i = 1, \ldots, j-1$.

A contradiction since
- Utility of a_j increases by $v_{a_j} - v_{a_i} > 0$, while
- utility of $a_1, a_2, \ldots, a_{j-1}$ and of the auctioneer is unchanged.

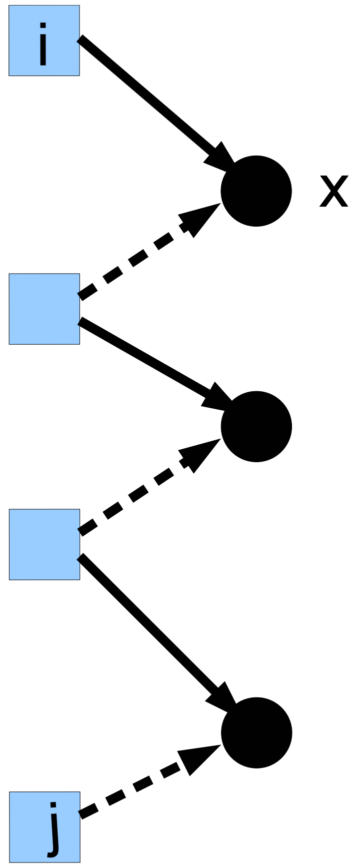
Proof of Pareto Optimality

Assume for contradiction there exists a forbidden alternating path ending at agent j in the final allocation.

Let $e = (i, x)$ be the earliest edge sold along the path. The edge was sold during some Sell(S) with $i \in S$.

$e = (i, x)$ contained in some S-AvoidMatching.

Lemma 2 *If there exists an alternating path from e to j in the final allocation (M, P) then there exists an alternating path from e to j in the S-Avoid matching when edge e is sold with same number of items sold to i and j.*



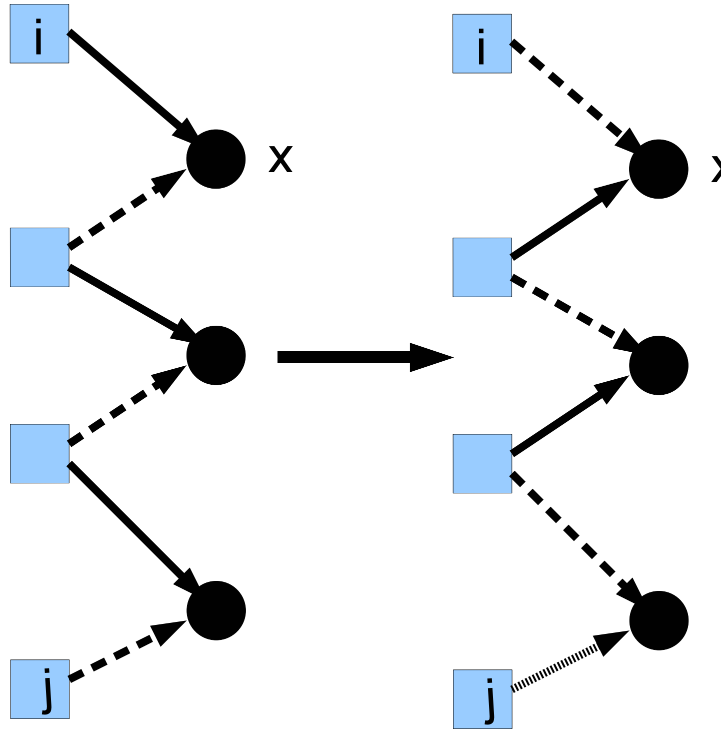
Proof of Pareto Optimality

Derive a contradiction either on the assignment of $e = (i, x)$ or on the existence of a forbidden alternating path.

Let $B(j)$ be the number of items assigned to j in the S-Avoid Matching.

Two cases:

1. $i \in V$. e is the last edge sold to i. Since $b_j \geq v_i$ we known $d_j > B(j)$. There exists an alternating path in the S-AvoidMatching formed by e and all edges sold after e that assigns one more item to j and one less item to i.



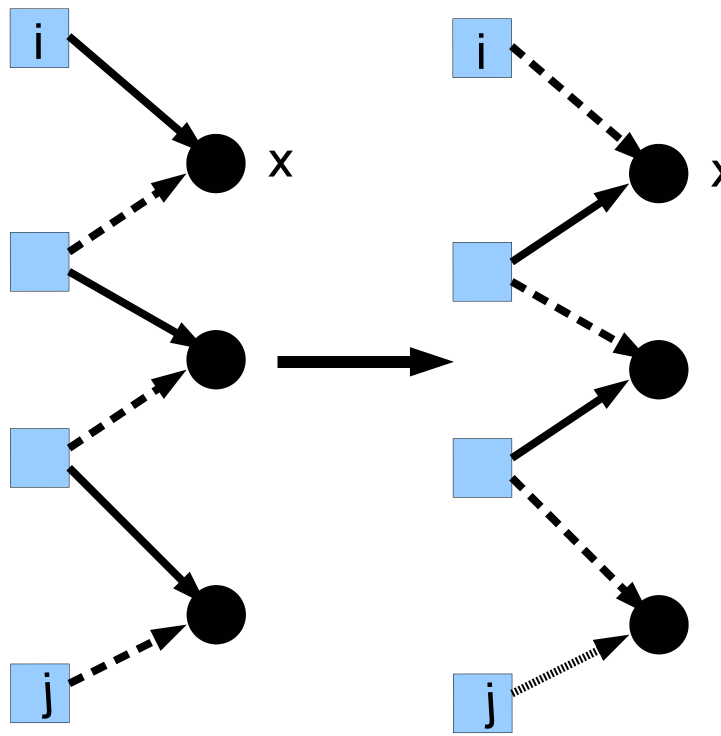
Proof of Pareto Optimality

2. $i \notin V$. Three cases

2.1 $d_j > B(j)$. There exists an S-Avoid matching that assigns one more item to j and one less item to i.

2.2 $d_j = B(j)$ and $d_j = D_j^+ < D_j$. The budget of agent j when e is sold is equal to $b_j = p \times D_j$. The remaining budget at the end of the auction is $\leq p < v_i$. The alternating path is not forbidden. A contradiction.

2.3 $d_j = B(j)$ and $d_j = D_j^+ = D_j$. A contradiction follows as in case [2.2].



We conclude that edge e cannot be sold or the alternating path is not forbidden.

Conclusions

Mapping the frontier

- If the sets of interest are public but budgets and valuations are private then no truthful Pareto-optimal auction is possible.
- If budgets are public but the sets of interest and the valuations are private then no truthful Pareto-optimal auction is possible.
- if budgets are public and private arbitrary valuations are allowed, no truthful and Pareto-optimal auction is possible (irrespective of computation time). This follows by simple reduction to the previous claim on private sets of interest.

Conclusion and Open problems

We present Pareto-optimal truthful combinatorial auction for single-valued agents with private valuations, public budgets and public interest sets.

- Randomization: Truthful in expectation?
- Envy-free allocations?
- Approximate social welfare
- Other mechanisms with different private/public partition?
- Position auctions with budgets?