3 Congestion Games

In this section, we consider a general class of resource allocation games, called congestion games.

**Definition 3.1 (Congestion model).** A congestion model $\mathcal{M} = (N, F, (X_i)_{i \in N}, (c_f)_{f \in F})$ is given by

- a set of players $N = [n]$;
- a set of facilities $F$;
- for every player $i \in N$, a set $X_i \subseteq 2^F$ of subsets of facilities in $F$;
- for every facility $f \in F$, a cost function $c_f : N \to \mathbb{R}$.

For every player $i \in N$, $X_i$ is the strategy set from which $i$ can choose. A strategy $x_i \in X_i$ is a subset of facilities; we think of $x_i$ as the facilities that player $i$ uses. Fix some strategy profile $x = (x_1, \ldots, x_n) \in X := X_1 \times \cdots \times X_n$. The cost incurred for the usage of facility $f \in F$ with respect to $x$ is defined as $c_f(n_f(x))$, where

$$n_f(x) := |\{i \in N : f \in x_i\}|$$

refers to the total number of players that use $f$.

**Definition 3.2 (Congestion game).** The congestion game corresponding to the congestion model $\mathcal{M} = (N, F, (X_i)_{i \in N}, (c_f)_{f \in F})$ is the strategic game $\Gamma = (N, (X_i)_{i \in N}, (c_i)_{i \in N})$, where every player $i \in N$ wants to minimize his cost

$$c_i(x) = \sum_{f \in x_i} c_f(n_f(x)).$$

(Equivalently, every player wants to maximize his utility $u_i = -c_i$.) The game is called symmetric if all players have the same strategy set, i.e., $X_i = Q$ for all $i \in N$ and some $Q \subseteq 2^F$.

**Example 3.1 (Atomic network congestion game).** The atomic network congestion game can be modeled as a congestion game: We are given a directed graph $G = (V, A)$, a single commodity $(s, t) \in V \times V$, and a cost function $c_a : N \to \mathbb{R}_+$ for every arc $a \in A$. Every player $i \in N$ wants to send one unit of flow from $s$ to $t$ along a single path. The set of facilities is

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*For a given set $S$, we use $2^S$ to refer to the power set of $S$, i.e., the set of all subsets of $S$.  

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$F := A$ and the strategy set $X_i$ of every player $i \in N$ is simply the set of all directed $s,t$-paths in $G$. (Note that the game is symmetric.) The goal of every player $i \in N$ is to choose a path $x_i \in X_i$ so as to minimize his cost

$$c_i(x) := \sum_{a \in x_i} c_a(n_a(x)),$$

where $n_a(x)$ refers to the total number of players using arc $a$. This example corresponds to a selfish routing game, where every player controls one unit of flow (i.e., we have atomic players) and has to route his flow unsplittably from $s$ to $t$.

### 3.1 Equivalence to Exact Potential Games

**Theorem 3.1.** Every congestion game $\Gamma = (N, (X_i)_{i \in N}, (c_i)_{i \in N})$ is an exact potential game.

**Proof.** Rosenthal’s potential function $\Phi : X \to \mathbb{R}$ is defined as

$$\Phi(x) := \sum_{f \in F} \sum_{k=1}^{n_f(x)} c_f(k). \quad (12)$$

We prove that $\Phi$ is an exact potential function for $\Gamma$. To see this, fix some $x \in X$, a player $i \in N$ and some $y_i \in X_i$. We have

$$\Phi(x_{-i}, y_i) = \sum_{f \in F} \sum_{k=1}^{n_f(x)} c_f(k) + \sum_{f \in y_i \setminus x_i} c_f(n_f(x) + 1) - \sum_{f \in x_i \setminus y_i} c_f(n_f(x))$$

$$= \Phi(x) + c_i(x_{-i}, y_i) - c_i(x).$$

Thus, $\Phi$ is an exact potential function. $\square$

By Theorem 2.3 it follows that every congestion game has the FIP and admits a pure Nash equilibrium. Moreover, by Theorem 2.7, the problem of computing a Nash equilibrium in congestion games is PLS-complete.

One can even prove the converse of Theorem 3.1: Every potential game can be transformed into an appropriate congestion game.

**Definition 3.3.** Let $\Gamma_1 = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ and $\Gamma_2 = (N, (Y_i)_{i \in N}, (v_i)_{i \in N})$ be two strategic games with player set $N := [n]$. $\Gamma_1$ and $\Gamma_2$ are isomorphic if for every player $i \in N$ there exists a bijection $\phi_i : X_i \to Y_i$ such that for all $x \in X$

$$u_i(x_1, \ldots, x_n) = v_i(\phi_1(x_1), \ldots, \phi_n(x_n)).$$

**Lemma 3.1.** Every coordination game is isomorphic to a congestion game.

**Proof.** Let $\Gamma_1 = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ be a coordination game with $n$ players that want to maximize a common utility function $u$. Introduce for every strategy profile $x \in X$ a unique facility $f(x)$ and let $F := \{f(x) : x \in X\}$ be the set of all facilities. We derive a congestion model $\mathcal{M} := (N, (Y_i)_{i \in N}, (c_f)_{f \in F})$ from $\Gamma_1$ as follows: The strategy set $Y_i$ of player $i \in N$ is defined as

$$Y_i := \{\phi_i(x_i) : x_i \in X_i\},$$
where \( \phi_i(x_i) \) refers to the set of all facilities \( f(x) \) that are associated with strategy profiles \( x \) in which player \( i \) chooses strategy \( x_i \), i.e., for every \( x_i \in X_i \)

\[
\phi_i(x_i) := \{ f(x_{-i}, x_i) : x_{-i} \in X_{-i} \}.
\]

Moreover, define the cost \( c_f(x) \) of a facility \( f(x) \in F \) as

\[
c_f(x)(k) := \begin{cases} 
  u(x) & \text{if } k = n \\
  0 & \text{otherwise}.
\end{cases}
\]

That is, facility \( f(x) \) has cost \( u(x) \) if every player in \( N \) uses this facility; otherwise, its cost is zero. Let \( \Gamma_2 \) be the congestion game that we obtain from \( \mathcal{M} \).

Fix some strategy profile \( x = (x_1, \ldots, x_n) \in X \) of \( \Gamma_1 \). We define the bijection \( \phi_i \) of every player \( i \in N \) as \( x_i \mapsto \phi_i(x_i) \). Note that for every strategy profile \( x \in X \)

\[
\bigcap_{i \in N} \phi_i(x_i) = f(x).
\]

Therefore there is exactly one facility, namely \( f(x) \), that is used by \( n \) players in \( \Gamma_2 \) with respect to the strategy profile \( (\phi_1(x_1), \ldots, \phi_n(x_n)) \). Thus, we have for every player \( i \in N \)

\[
u(x) = c_f(x)(n) = c_i(\phi_1(x_1), \ldots, \phi_n(x_n)),
\]

which shows that \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic.

Lemma 3.2. Every dummy game is isomorphic to a congestion game.

The proof of this lemma is slightly more complicated but similar in flavor to the one above and omitted here.

Theorem 3.2. Every potential game is isomorphic to a congestion game.

Proof. By Theorem 2.6, we can decompose the potential game into a coordination game and a dummy game. Apply the above theorems to obtain congestion games \( \Gamma_1 \) and \( \Gamma_2 \) (with disjoint facility sets) that are isomorphic to the coordination and the dummy game, respectively. We can then construct an isomorphic congestion game by taking the union of the facility sets and players' strategies (strategy-wise) of \( \Gamma_1 \) and \( \Gamma_2 \).

3.2 Price of Anarchy

Define the social cost of a strategy profile \( x \in X \) as the total cost of all players, i.e.,

\[
c(x) := \sum_{i \in N} c_i(x) = \sum_{f \in F} n_f(x)e_f(n_f(x)).
\]

As before, we define \( \text{opt}(\Gamma) := \min_{x \in X} c(x) \) as the optimal cost for \( \Gamma \).

We derive an upper bound on the price of anarchy for congestion games with respect to the social cost function \( c \) defined above. Here we only consider the case that the cost of every facility \( f \in F \) is given as \( c_f(k) = k \). The proof extends to arbitrary linear latency functions.
Theorem 3.3. Let $\mathcal{M} = (N,(X_i)_{i \in N},(c_i)_{i \in N})$ be a congestion model with linear latency functions $c_f(k) = k$ for every $f \in F$ and let $\Gamma = (N,(X_i)_{i \in N},(u_i)_{i \in N})$ be the corresponding congestion game. The price of anarchy is at most $5/2$.

We will use the following fact to prove this theorem (whose proof we leave as an exercise):

Fakt 3.1. Let $\alpha$ and $\beta$ be two non-negative integers. Then
\[
\alpha(\beta + 1) \leq \frac{5}{3} \alpha^2 + \frac{1}{3} \beta^2.
\]

Proof of Theorem 3.3. Let $x$ be a Nash equilibrium and $x^*$ be an optimal strategy profile minimizing $c$. Since $x$ is a Nash equilibrium, the cost of every player $i \in N$ does not decrease if he deviates to his optimal strategy $x_i^*$, i.e.,
\[
c_i(x) \leq c_i(x_{-i},x_i^*) = \sum_{f \in x_i^*} c_f(n_f(x_{-i},x_i^*)) = \sum_{f \in x_i^*} n_f(x_{-i},x_i^*) \leq \sum_{f \in x_i^*} n_f(x) + 1,
\]
where the last inequality follows since player $i$ increases the number of players on each $f \in x_i^*$ by at most 1 with respect to $n_f(x)$. Summing over all players, we obtain
\[
c(x) = \sum_{i \in N} c_i(x) \leq \sum_{i \in N} \sum_{f \in x_i^*} n_f(x) + 1 = \sum_{f \in F} n_f(x^*)(n_f(x) + 1).
\]
Using Fact 3.1, we therefore obtain
\[
c(x) \leq \sum_{f \in F} n_f(x^*)(n_f(x) + 1) \leq \frac{5}{3} \sum_{f \in F} (n_f(x^*))^2 + \frac{1}{3} \sum_{f \in F} (n_f(x))^2 = \frac{5}{3} c(x^*) + \frac{1}{3} c(x),
\]
where the last equality follows from $c_f(k) = k$ for every $f \in F$ and the definition of $c$. We conclude that $c(x) \leq \frac{5}{2} c(x^*)$. \qed