Altruism and Spite in Games

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Summer School on Algorithmic Game Theory
Samos, July 14–21, 2012
Motivation
Situations of strategic interaction

**Viewpoint:** many real-world problems are complex and distributed in nature

- involve several independent decision makers (*players*)
- decision makers attempt to achieve their own goals (*selfish*)

**Examples:** network routing, Internet applications, auctions, ...

**Phenomenon:** strategic behavior leads to outcomes that are suboptimal for society as a whole

**Need:** gain fundamental understanding of the effect of strategic interaction in such applications
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Contributions of AGT

Algorithmic game theory:

- mathematical toolbox to study such situations
- focus on algorithmic and computational issues
- advanced our understanding of several phenomena

Some contributions:

- complexity of reaching stable outcomes (PPAD, PLS)
- inefficiency of equilibria (price of anarchy, price of stability)
- mechanisms that steer selfish behavior into more favorable outcomes (coordination mechanisms)
- ...
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Criticism

1 Most studies consider Nash equilibria as solution concept
   Assumption that computationally bounded players can reach such outcomes is questionable!
   ⇒ study inefficiency of more permissive solution concepts (correlated, coarse equilibria) and natural response dynamics (“robust price of anarchy”)

2 Self-interest hypothesis: every player makes his choice based on purely selfish motives
   Assumption is at odds with other-regarding preferences observed in practice (altruism, spite, fairness).
   ⇒ model such alternative behavior and study its impact on the outcomes of games
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⇒ model such alternative behavior and study its impact on the outcomes of games
Example: Single-item auction

Setting:
- \( n \) bidders are interested in receiving a single item
- every bidder \( i \) has a private value \( v_i \) and a bid \( b_i \)
- auctioneer: determines who obtains the item and a price \( p \)
- utility of bidder \( i \) is \( v_i - p \) if \( i \) gets the item and 0 otherwise

Theory: second-price auction ensures that bidding truthfully \( (b_i = v_i) \) is a dominant strategy for every player \( i \)

Observations: bidders get carried away and submit bids that significantly exceed their value for the item (bidding frenzy) [Morgan et al. '03]

Explanation: bidders care negatively about the surplus of their competitors and might therefore be spiteful
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[Morgan et al. '03]
Example: Public goods

Setting:
- \( n \) players are interested in a public good
- every player \( i \) chooses a contribution \( s_i \in [0, b] \)
- total contribution is doubled and distributed among players
- payoff of player \( i \): \( p_i = b - s_i + \frac{2}{n} \sum_{j=1}^{n} s_j \)

Theory: \( s_i = 0 \) is dominant strategy for every player \( i \) (each player will try to free-ride)

Observations: individual contribution of each player typically ranges between 40% and 60% of \( b \)

Explanation: players are partially altruistic and contribute to the public good even if they run the risk that others might free-ride
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Question:
Can we gain an accurate understanding of the impact of such “social” interactions in games?
Overview

Motivation

Part I: Altruistic games
• modeling altruistic behavior in games
• inefficiency of equilibria

Part II: Smoothness technique
• smoothness and robust price of anarchy
• adaptations to altruistic games

Part III: Results in a nutshell
• linear congestion games
• fair cost-sharing games
• valid utility games

Concluding remarks
Altruistic Games
A cost minimization game \( G = (N, (S_i)_{i \in N}, (C_i)_{i \in N}) \) is a finite strategic game given by

- set of players \( N = [n] \)
- set of strategies \( S_i \) for every player \( i \in N \)
- cost function \( C_i : S_1 \times \cdots \times S_n \to \mathbb{R} \)

Every player \( i \in N \) chooses his strategy \( s_i \in S_i \) so as to minimize his individual cost \( C_i(s_1, \ldots, s_n) \)

Let \( S = S_1 \times \cdots \times S_n \) be the set of strategy profiles.

Social cost of strategy profile \( s = (s_1, \ldots, s_n) \in S \) is

\[
C(s) = \sum_{i \in N} C_i(s)
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Equilibrium concepts

Nash equilibrium: $s = (s_1, \ldots, s_n) \in S$ is a pure Nash equilibrium (PNE) if no player has an incentive to unilaterally deviate

$$\forall i \in N : C_i(s_i, s_{-i}) \leq C_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i$$

$(s_{-i} \text{ refers to } (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n))$

More general solution concepts:

• mixed Nash equilibrium (MNE)
• correlated equilibrium (CE)
• coarse correlated equilibrium (CCE)
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Example: Congestion game

\[ n = 10 \]
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Nash equilibrium: \( C(s) = 100 \)
Example: Congestion game

\[ n = 10 \]

social optimum: \( C(s^*) = 75 \)
Example: Congestion game

\[ n = 10 \]

\[
\text{inefficiency: } \frac{C(s)}{C(s^*)} = \frac{100}{75} = \frac{4}{3}
\]
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Nash equilibrium: \[ C(s) = 91 \]
Example: Congestion game

\[ n = 10 \]

\[ \text{inefficiency: } \frac{C(s)}{C(s^*)} = \frac{91}{75} \approx 1.21 \]
Inefficiency of equilibria

Let $s^*$ be a strategy profile that minimizes the social cost $C(s)$.

**Price of anarchy:** worst-case inefficiency of equilibria

$$POA(G) = \max_{s \in PNE(G)} \frac{C(s)}{C(s^*)}$$

[Koutsoupias, Papadimitriou, STACS '99]

**Price of stability:** best-case inefficiency of equilibria

$$POS(G) = \min_{s \in PNE(G)} \frac{C(s)}{C(s^*)}$$

[Schulz, Moses, SODA '03]

**Remark:** definitions extend to other solution concepts (such as MNE, CE, CCE) in the obvious way
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Altruistic extensions of strategic games

base game \( G = (N, (S_i)_{i \in N}, (C_i)_{i \in N}) \)
Altruistic extensions of strategic games

**base game** $G = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$

altruism level $\alpha_i \in [0, 1]$ for every player $i \in N$
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**base game** $G = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$

altruism level $\alpha_i \in [0, 1]$ for every player $i \in N$

**altruistic extension** $G^\alpha = (N, (S_i)_{i \in N}, (C_i^\alpha)_{i \in N})$ of $G$ with

$$C_i^\alpha(s) = (1 - \alpha_i)C_i(s) + \alpha_iC(s)$$
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\[
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\( \alpha_i = 0 \) \hspace{1cm} \alpha_i \hspace{1cm} \alpha_i = 1 \\
egoist \hspace{1cm} \alpha_i\text{-altruist} \hspace{1cm} altruist
Some remarks

Viewpoint:

- $C_i^\alpha$ is the perceived cost of $i$ (encodes $i$’s altruistic behavior)
- outcome is determined by players minimizing their perceived costs
- $C_i$ is the actual cost that player $i$ contributes to the social cost
  ⇒ consider unaltered social cost function

$$C(s) = \sum_{i \in N} C_i(s)$$

Advantages of this approach:

- altruistic extension contains the base game as a special case
- stay in the domain of the base game (here: strategic games)
- can use standard solution concepts, methodologies, etc.
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Other models

1. \( C_i^\alpha(s) = (1 - \alpha)C_i(s) + \alpha C(s) \) [Chen et al., WINE '11]
2. \( C_i^\beta(s) = (1 - \beta)C_i(s) + \frac{\beta}{n} C(s) \) [Chen, Kempe, EC '08]
3. \( C_i^\xi(s) = (1 - \xi)C_i(s) + \xi \sum_{j \neq i} C_j(s) \) [Caragiannis et al., TGC '10]
4. \( C_i^\alpha(s) = C_i(s) + \alpha C(s) \) [Apt, Schäfer '12]
5. . .

**Observation:** above models are equivalent for suitable transformations of the altruism parameters
Example: Altruistic congestion game

PNE conditions: $s$ is Nash equilibrium of $G^\alpha$ if for every $i \in N$:

$$(1 - \alpha)C_i(s_i, s_{-i}) + \alpha C(s, s_{-i}) \leq (1 - \alpha)C_i(s', s_{-i}) + \alpha C(s', s_{-i})$$
Example: Altruistic congestion game

\[ \alpha = 0 \]

**PNE conditions:** \( s \) is Nash equilibrium of \( G^\alpha \) if for every \( i \in N \):

\[
(1 - \alpha)C_i(s_i, s_{-i}) + \alpha C(s_i, s_{-i}) \leq (1 - \alpha)C_i(s'_i, s_{-i}) + \alpha C(s'_i, s_{-i})
\]

\[ \Leftrightarrow (1 - \alpha)10 + \alpha (10 \cdot 10) \leq (1 - \alpha)10 + \alpha (9 \cdot 9 + 10) \]

\[ \Leftrightarrow \alpha \leq 0 \]
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$\Leftrightarrow (1 - \alpha)9 + \alpha(9 \cdot 9 + 10) \leq (1 - \alpha)10 + \alpha(8 \cdot 8 + 2 \cdot 10)$

$\Leftrightarrow \alpha \leq 1/8$
Example: Altruistic congestion game

\[ 0 < \alpha \leq \frac{1}{8} \]

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\[ \frac{1}{8} < \alpha \leq \cdot \]

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$\Leftrightarrow (1 - \alpha)8 + \alpha(8 \cdot 8 + 2 \cdot 10) \leq (1 - \alpha)10 + \alpha(7 \cdot 7 + 3 \cdot 10)$

$\Leftrightarrow \alpha \leq 2/7$
Example: Altruistic congestion game

\[ \frac{1}{8} < \alpha \leq \frac{2}{7} \]

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\]

\[ \iff (1 - \alpha)8 + \alpha(8 \cdot 8 + 2 \cdot 10) \leq (1 - \alpha)10 + \alpha(7 \cdot 7 + 3 \cdot 10) \]

\[ \iff \alpha \leq \frac{2}{7} \]
Example: Altruistic congestion game

$\frac{2}{7} < \alpha \leq \frac{3}{6}$

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Example: Altruistic congestion game

\[
\frac{3}{6} < \alpha \leq \frac{4}{5}
\]

\[
\begin{array}{c}
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Example: Altruistic congestion game

\[ \alpha > \frac{4}{5} \]

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\]
Example: Price of anarchy

$POA$
Related Work

[Chen and Kempe, EC ’08]: altruism and spite in non-atomic network routing games

- uniform altruism: \( \text{POA} \leq \frac{1}{\beta} \)
- uniform spite/altruism, affine latencies: \( \text{POA} \leq \frac{4}{3+2\beta+\beta^2} \)
- non-uniform altruism, parallel links: \( \text{POA} \leq \frac{1}{\bar{\beta}} \)

[Hoefer and Skopalik, ESA ’09]: uniform altruism in congestion games

- existence of pure NE (exist for affine cost functions)
- convergence of sequential best-response dynamics
Related Work

[Chen and Kempe, EC ’08]: altruism and spite in non-atomic network routing games

- uniform altruism: $\text{POA} \leq 1/\beta$
- uniform spite/altruism, affine latencies: $\text{POA} \leq \frac{4}{3+2\beta+\beta^2}$
- non-uniform altruism, parallel links: $\text{POA} \leq 1/\bar{\beta}$

[Hoefer and Skopalik, ESA ’09]: uniform altruism in congestion games

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Related Work

[Caragiannis et al., TGC ’10]: uniform altruism in congestion and load balancing games
  • derive bounds on the POA for affine cost functions
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  • players are (completely) altruistic towards “friends”
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Smoothness Technique
Smoothness

A strategic game $G$ is $(\lambda, \mu)$-smooth if for any two strategy profiles $s, s^* \in S$

$$\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) \leq \lambda C(s^*) + \mu C(s).$$

[Roughgarden, STOC '09]

The robust price of anarchy of a game $G$ is defined as

$$RPOA(G) = \inf \left\{ \frac{\lambda}{1 - \mu} : G \text{ is } (\lambda, \mu)\text{-smooth with } \mu < 1 \right\}.$$
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Consequences in a nutshell

Theorem

Let $G$ be a game with robust price of anarchy $RPOA(G)$.

1. The price of anarchy of coarse correlated equilibria of $G$ is at most $RPOA(G)$.

2. The average cost of a sequence of outcomes of $G$ with vanishing average external regret approaches $RPOA(G) \cdot C(s^*)$.

3. If $G$ admits an exact potential function, then best-response dynamics quickly reach an outcome of cost at most $RPOA(G) \cdot C(s^*)$.

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[Roughgarden, STOC ’09]
Suppose \( s = (s_1, \ldots, s_n) \in S \) is a pure Nash equilibrium. Fix an optimal strategy profile \( s^* = (s^*_1, \ldots, s^*_n) \in S \). Then

\[
C(s) = \sum_{i \in N} C_i(s_i, s_{-i}) \leq \sum_{i \in N} C_i(s^*_i, s_{-i}) \quad \text{(exploiting PNE conditions)}
\]

\[
\leq \lambda C(s^*) + \mu C(s) \quad \text{(exploiting \((\lambda, \mu)\)-smoothness)}
\]

By rearranging terms, we obtain

\[
\frac{C(s)}{C(s^*)} \leq \frac{\lambda}{1 - \mu} \quad \text{and thus} \quad POA \leq \frac{\lambda}{1 - \mu}.
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Suppose $s = (s_1, \ldots, s_n) \in S$ is a pure Nash equilibrium. Fix an optimal strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$. Then

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Glimpse: Pure price of anarchy

Suppose \( s = (s_1, \ldots, s_n) \in S \) is a pure Nash equilibrium. Fix an optimal strategy profile \( s^* = (s_1^*, \ldots, s_n^*) \in S \). Then

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\]
Let $\sigma^1, \ldots, \sigma^T$ be a sequence of probability distributions over outcomes of $G$ in which every player experiences vanishing average external regret, i.e., for every $i \in N$ and $s'_i \in S_i$:

$$E \left[ \sum_{t=1}^{T} C_i(s^t) \right] \leq E \left[ \sum_{t=1}^{T} C_i(s'_i, s^t_{\neg i}) \right] + o(T). \quad (*)$$

→ no-regret algorithms

[Hart and Mas-Colell '00]

Exploiting the smoothness condition and $(*)$, it follows that the average cost of this sequence satisfies

$$\frac{1}{T} \sum_{t=1}^{T} E \left[ C(s^t) \right] \leq RPOA(G) \cdot C(s^*) \quad \text{as} \quad T \to \infty.$$
Let $\sigma^1, \ldots, \sigma^T$ be a sequence of probability distributions over outcomes of $G$ in which every player experiences vanishing average external regret, i.e., for every $i \in N$ and $s'_i \in S_i$:

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→ no-regret algorithms [Hart and Mas-Colell ’00]

Exploiting the smoothness condition and $(\ast)$, it follows that the average cost of this sequence satisfies

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Glimpse: No-regret sequences

Let $\sigma^1, \ldots, \sigma^T$ be a sequence of probability distributions over outcomes of $G$ in which every player experiences vanishing average external regret, i.e., for every $i \in N$ and $s'_i \in S_i$:

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$\rightarrow$ no-regret algorithms

Exploiting the smoothness condition and ($\ast$), it follows that the average cost of this sequence satisfies

$$\frac{1}{T} \sum_{t=1}^{T} E \left[ C(s^t) \right] \leq RPOA(G) \cdot C(s^*) \text{ as } T \rightarrow \infty.$$
For a given strategy profile $s \in S$, define

$$C_{-i}(s) = \sum_{j \neq i} C_j(s).$$

An altruistic game $G^\alpha$ is $(\lambda, \mu, \alpha)$-smooth if for any two strategy profiles $s, s^* \in S$

$$\sum_{i=1}^{n} C_i(s^*_i, s_{-i}) + \alpha_i(C_{-i}(s^*_i, s_{-i}) - C_{-i}(s)) \leq \lambda C(s^*) + \mu C(s).$$

Define the robust price of anarchy of an altruistic game $G^\alpha$ as

$$RPOA(G^\alpha) = \inf \left\{ \frac{\lambda}{1 - \mu} : G^\alpha \text{ is } (\lambda, \mu, \alpha)\text{-smooth with } \mu < 1 \right\}.$$
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Adapted smoothness notion

For a given strategy profile $s \in S$, define

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Define the **robust price of anarchy** of an altruistic game $G^\alpha$ as

$$RPOA(G^\alpha) = \inf \left\{ \frac{\lambda}{1 - \mu} : G^\alpha \text{ is } (\lambda, \mu, \alpha)\text{-smooth with } \mu < 1 \right\}.$$
Implications

Can generalize most of the results of [Roughgarden, STOC ’09] to altruistic extensions of games:

**Theorem**

Suppose the robust price of anarchy of $G^\alpha$ is $RPOA(G^\alpha)$.

1. The price of anarchy of coarse correlated equilibria of $G^\alpha$ is at most $RPOA(G^\alpha)$.

2. The average cost of a sequence of outcomes of $G^\alpha$ with vanishing average external regret approaches $RPOA(G^\alpha) \cdot C(s^*)$.

3. If $G^\alpha$ admits an exact potential function, then best-response dynamics quickly reach an outcome of cost at most $RPOA(G^\alpha) \cdot C(s^*)$. 
Results in a Nutshell

joint work:

Po-An Chen, Bart de Keijzer and David Kempe
Altruistic congestion games

Results in a nutshell:

1 The robust price of anarchy of $\alpha$-altruistic linear congestion games is at most

$$\frac{5 + 2\hat{\alpha} + 2\check{\alpha}}{2 - \hat{\alpha} + 2\check{\alpha}},$$

where $\hat{\alpha}$ and $\check{\alpha}$ are the maximum and minimum altruism levels, respectively.

2 This bound specializes to $\frac{5 + 4\alpha}{2 + \alpha}$ for uniformly $\alpha$-altruistic congestion games and is tight even for pure NE.

3 The pure price of stability of uniformly $\alpha$-altruistic congestion games is at most $\frac{2}{1 + \alpha}$.

[Caragiannis et al., TGC '10]
Altruistic congestion games

Results in a nutshell:

1. The **robust price of anarchy** of $\alpha$-altruistic linear congestion games is at most

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where $\hat{\alpha}$ and $\check{\alpha}$ are the maximum and minimum altruism levels, respectively.

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2. This bound specializes to $\frac{5 + 4\alpha}{2 + \alpha}$ for uniformly $\alpha$-altruistic congestion games and is tight even for pure NE. [Caragiannis et al., TGC ’10]

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Bounds for uniform players

![Graph showing bounds for uniform players.]
Bounds for uniform players

Guido Schäfer

Altruism and Spite in Games
The pure price of anarchy of uniformly $\alpha$-altruistic extensions of symmetric singleton linear congestion games is $\frac{4}{3+\alpha}$.

[Caragiannis et al., TGC '10]

The mixed price of anarchy of $\alpha$-altruistic extensions of symmetric singleton linear congestion games is at least 2.

The pure price of anarchy of $\alpha$-altruistic extensions of symmetric singleton linear congestion games with $\alpha \in \{0, 1\}^n$ is at most $\frac{4-2\bar{\alpha}}{3-\bar{\alpha}}$, where $\bar{\alpha}$ is the fraction of purely altruistic players.
Altruistic singleton congestion games

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The pure price of anarchy of uniformly $\alpha$-altruistic extensions of symmetric singleton linear congestion games is $\frac{4}{3+\alpha}$. [Caragiannis et al., TGC ’10]

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Altruistic cost-sharing games

**Fair cost-sharing game:** players choose facilities and the cost of each selected facility is evenly shared among the players using it.

**Results in a nutshell:**

1. The robust price of anarchy of $\alpha$-altruistic cost-sharing games is $\frac{n}{1-\alpha}$ (with $n/0 = \infty$).

2. This bound is tight for the pure price of anarchy of uniformly $\alpha$-altruistic extensions of network cost-sharing games.

3. The pure price of stability of uniformly $\alpha$-altruistic cost-sharing games is at most $(1 - \alpha)H_n + \alpha$. 
Valid utility games: model “two-sided market games” such as the facility location game

Results in a nutshell:

1. The robust price of anarchy of $\alpha$-altruistic extensions of valid utility games is 2, independent of the altruism level distribution.

2. This bound is tight for the pure price of anarchy of $\alpha$-altruistic extensions of valid utility games.
Concluding remarks
Concluding remarks

Summary:
- initiated the study of the impact of altruism in strategic games
- smoothness framework to bound the inefficiency of altruistic games
- approach is powerful enough to derive tight bounds on the robust price of anarchy of altruistic extensions of congestion games, cost-sharing games and valid utility games

Conclusions:
- altruistic behavior may lead to an increase in the inefficiency
- not a universal phenomenon though: the price of anarchy may decrease (singleton congestion games) or remain the same (valid utility games)
Ongoing and Future Work

Ongoing work: together with Krzysztof Apt

- use similar idea to define an alternative inefficiency measure

Question: How much altruism does one have to add to a game such that the price of stability becomes 1?

Selfishness level: smallest value $\alpha$ such that the $\alpha$-altruistic extension has a price of stability of 1

Results:
- prove general characterization result
- invariant under linear transformations
- ordinal potential games have finite selfishness level
- determine selfishness level of several classical games

→ see next talk by Krzysztof!
Thank you!