Coordination Games on Graphs
(Extended Abstract)

Krzysztof R. Apt[1,2], Mona Rahn[1], Guido Schäfer[1,3], and Sunil Simon[4]

[1] Centre for Mathematics and Computer Science (CWI)
[2] ILLC, University of Amsterdam, The Netherlands
[3] VU University Amsterdam, The Netherlands
[4] IIT Kanpur, Kanpur, India

Abstract. We introduce natural strategic games on graphs, which capture the idea of coordination in a local setting. We show that these games have an exact potential and have strong equilibria when the graph is a pseudoforest. We also exhibit some other classes of graphs for which a strong equilibrium exists. However, in general strong equilibria do not need to exist. Further, we study the (strong) price of stability and anarchy. Finally, we consider the problems of computing strong equilibria and of determining whether a joint strategy is a strong equilibrium.

1 Introduction

Motivation. In game theory coordination games are used to model situations in which players are rewarded for agreeing on a common strategy, e.g., by deciding on a common technological or societal standard. In this paper we propose and study a very simple and natural class of coordination games, which we call *coordination games on graphs*: We are given an undirected graph the nodes of which correspond to the players of the game. Each player chooses a colour from a set of colours available to her. The payoff of a player is the number of neighbours who choose the same colour.

This model captures situations in which players have to choose between multiple competing providers offering the same service (or product), such as peer-to-peer networks, social networks, photo sharing platforms, mobile phone providers, etc. In these applications the benefit of a player subscribing to a specific provider increases as certain other players (e.g., friends, relatives, etc.) opt for the same provider. As a consequence, players have an interest to coordinate their choices in order to maximize their benefit.

Our contributions. The focus of our investigations is on coalitional equilibria in coordination games on graphs. Recall that in a *strong equilibrium* (*k-equilibrium*) no coalition of players (of size at most k) can deviate so that every player of the coalition strictly improves her payoff. Our main contributions are as follows:

1. *Existence.* We show that k-equilibria for $k \geq 5$ do not exist in general. Further, we identify several graph structural properties that guarantee the

existence of strong equilibria and also ensure that every sequence of profitable joint deviations terminates. Further, we show that 2-equilibria (and hence Nash equilibria) always exist.

2. *Inefficiency.* We derive almost matching lower and upper bounds of $2\frac{n-1}{k-1}-1$ and $2\frac{n-1}{k-1}$, respectively, on the k-price of anarchy. Further, the strong price of anarchy is exactly 2. We also provide conditions on the graph guaranteeing that the strong price of stability is 1.
3. *Computability.* We prove that the problem of deciding whether a given joint strategy is a k-equilibrium is co-NP-complete. On the positive side, for certain graph classes the decision problem is in P and we can efficiently compute a strong equilibrium.

Related work. Given their simplicity, it is not surprising that coordination games on graphs are related to various well-studied types of games. Due to lack of space, we only mention the most relevant references below.

First, they are similar to *additively separable hedonic games* [6, 4]. Here players are nodes on a weighted graph and form coalitions. The payoff of a node is the total weight of all incident edges to neighbours in the same coalition. These games were originally introduced in a cooperative setting but have more recently also been investigated in a strategic setting, with a particular focus on computational issues (e.g., [8]); for a survey see [7]. Aziz and Brandl [3] study the existence of strong equilibria in these games. Despite the similarity between these games and our coordination games, in the former every player can choose to enter every possible coalition and hence they do not generalize the games we study here.

Next, coordination games on graphs are *polymatrix games* (see [10]). Recall that this is a finite strategic game in which the payoff for each player is the sum of the payoffs obtained from the individual 2-player games she plays with each other player separately. Hoefer [9] studied clustering games that are also polymatrix games based on graphs. However, in his setup each player has the same set of strategies, so the resulting games are not comparable to ours.

When the graph is complete, coordination games on graphs are special cases of congestion games with monotone increasing utility functions. Rozenfeld and Tennenholtz [12] give a characterization of the existence of strong equilibria for these games. Bilò et al. [5] study congestion games in which the players form a (possibly directed) influence graph, focusing on the existence of Nash equilibria. However, because the latency functions are assumed to be increasing in the number of players, these games do not cover the games we study here.

As for the solution concepts, strong equilibria were introduced in [2]; the strong price of anarchy was defined in [1]; and finally, exact potentials were introduced in [11].

2 Coordination games on graphs

We next introduce some standard notion and define our coordination games on graphs.

A ***strategic game*** $\mathcal{G} := (N, (S_i)_{i \in N}, (p_i)_{i \in N})$ consists of a set $N := \{1, \ldots, n\}$ of $n > 1$ players and a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ for each player $i \in N$. We denote $S_1 \times \cdots \times S_n$ by S and call each element $s \in S$ a ***joint strategy***.

We call a non-empty subset $K := \{k_1, \ldots, k_m\}$ of N a ***coalition***. Given a joint strategy s we abbreviate the sequence $(s_{k_1}, \ldots, s_{k_m})$ of strategies to s_K; we also write (s_K, s_{-K}) instead of s. Given two joint strategies s and s' and a coalition K, we say that the players in K can profitably deviate from s to s' if $s' = (s'_K, s_{-K})$ and $p_i(s') > p_i(s)$ for every player $i \in K$. A joint strategy s a ***k-equilibrium*** $(1 \leq k \leq n)$ if no coalition of at most k players that can profitably deviate from s. Using this definition, a ***Nash equilibrium*** is a 1-equilibrium and a ***strong equilibrium*** is an n-equilibrium.

Given a joint strategy s, its ***social welfare*** is defined as $SW(s) = \sum_{i \in N} p_i(s)$. When the social welfare of s is maximal we call s a ***social optimum***. Given a finite game that has a k-equilibrium its ***k-price of anarchy (resp. stability)*** is the ratio $SW(s)/SW(s')$, where s is a social optimum and s' is a k-equilibrium with the lowest (resp. highest) social welfare. In the case of division by zero, we interpret the outcomes as ∞. The ***(strong) price of anarchy*** refers to the k-price of anarchy with $k = 1$ (resp. $k = n$). The ***(strong) price of stability*** is defined analogously.

A ***coalitional improvement path***, in short a ***c-improvement path***, is a maximal sequence $(s^1, s^2, \ldots)$ of joint strategies such that for every $k > 1$ there is a coalition K such that s^k is a profitable deviation of the players in K from s^{k-1}. Clearly, if a c-improvement path is finite, its last element is a strong equilibrium. We say that $\mathcal{G}$ has the ***finite c-improvement property*** (***c-FIP***) if every c-improvement path is finite.

Our ***coordination games on graphs*** are defined as follows: We are given a finite set of ***colours*** M, an undirected graph $G = (V, E)$ without self-loops, and a ***colour assignment*** A. The latter is a function that assigns to each node $i \in V$ a non-empty set of colours $A_i \subseteq M$. Let N_i denote the set of all ***neighbours*** of node i, i.e., $N_i = \{j \in V \mid \{i, j\} \in E\}$. We define a strategic game $\mathcal{G}(G, A)$ as follows:

- the players are identified with the nodes, i.e., $N = V$,
- the set of strategies of player i is the set of colours $A_i \subseteq M$,
- the payoff function of player i is $p_i(s) = |\{j \in N_i \mid s_i = s_j\}|$.

So each node simultaneously chooses a colour and the payoff to the node is the number of neighbours who chose the same colour. Subsequently, we refer to these games simply as ***coordination games***.

3 Existence

We identify several graph structural properties that guarantee the existence of strong equilibria in coordinate games. Most of our existence results are based on the following key lemma.

Given a set of nodes K, we denote by $G[K]$ the subgraph of G induced by K and by $E[K]$ the set of edges in E that have both endpoints in K. Recall that an edge set $F \subseteq E[K]$ is a ***feedback edge set*** of $G[K]$ if the graph $(K, E[K] \setminus F)$ is acyclic. Given a joint strategy s we denote by E_s^+ the set of edges $\{i, j\} \in E$ such that $s_i = s_j$.

Lemma 1 (Key lemma). *Suppose a coalition K can profitably deviate from s to s'. Let F be a feedback edge set of $G[K]$. Then*

$$SW(s') - SW(s) > 2|F \cap E_s^+| - 2|F \cap E_{s'}^+|.$$

Using this key lemma, we can derive several existence results.

Theorem 1. *Every c-improvement path in which deviations of coalitions of size at most 2 are allowed is finite. In particular, 2-equilibria (and thus Nash equilibria) always exist.*

Theorem 2. *Every coordination game with at most 2 colours has the c-FIP.*

For a colour $x \in M$ let $V_x = \{i \in V \mid x \in A_i\}$ be the set of nodes that can choose x. We call G a ***colour forest*** (with respect to A) if $G[V_x]$ is a forest for all $x \in M$.

Theorem 3. *Every coordination game on a colour forest has the c-FIP.*

Recall that a ***pseudoforest*** is a graph in which every connected component contains at most one cycle.

Theorem 4. *Every coordination game on a pseudoforest has the c-FIP.*

Theorem 5. *Let G be such that cycles are pairwise edge-disjoint. Let k be the minimum length of a cycle in G. Then every coalitional improvement path of coalitions of size at most k is finite. Hence a k-equilibrium exists.*

We also establish the c-FIP property for some additional classes of coordination games. We call a coordination game on a graph G ***uniform*** if for every joint strategy s and for every edge $\{i, j\} \in E$ it holds: if $s_i = s_j$ then $p_i(s) = p_j(s)$.

Theorem 6. *Every uniform coordination game has the c-FIP.*

A class of coordination games that we can capture by Theorem 6 is as follows. We say that G is ***colour complete*** (with respect to A) if each component of $G[V_x]$ is complete for every $x \in M$. In particular, every complete graph is colour complete.

Corollary 1. *Every coordination game on a colour complete graph has the c-FIP.*

Above we identified sufficient properties of graphs that ensure the existence of strong equilibria. In general, coordination games may not admit strong equilibria. To see this, consider the coordination game depicted in Figure 1. It is not hard to verify that for every joint strategy there is an improving move by a coalition of size at most 5. Thus, 5-equilibria (and hence strong equilibria) do not need to exist.

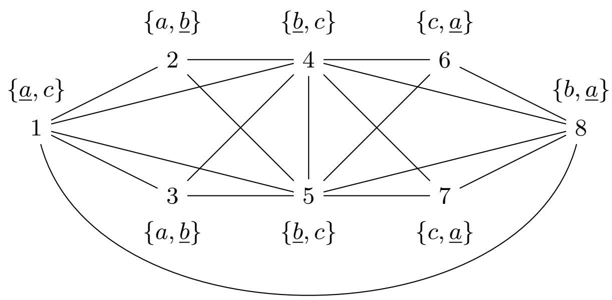

Fig. 1. A coordination game with three colours that does not admit a 5-equilibrium.

4 Inefficiency

The next two theorems summarize our results on the inefficiency of k-equilibria.

Theorem 7. *The price of anarchy of coordination games is ∞; the strong price of anarchy is 2. Further, for all $k \in \{2, \dots, n-1\}$, the k-price of anarchy for coordination games is between $2\frac{n-1}{k-1} - 1$ and $2\frac{n-1}{k-1}$.*

Theorem 8. *The strong price of stability is 1 in each of the following cases: G is a pseudoforest; G is a colour forest; there are only two colours.*

5 Computation

In general it is hard to decide whether a given joint strategy is a k-equilibrium.

Theorem 9. *Given a joint strategy s of a coordination game and $k \in \{1, \dots, n\}$, it is co-NP-complete to decide whether s is a k-equilibrium.*

However, we can derive positive results for colour forests and pseudoforests.

Theorem 10. *Let G be a colour forest. Then there exists an algorithm that determines in polynomial time whether a given joint strategy is a k-equilibrium. Further, a strong equilibrium can be computed in polynomial time.*

Theorem 11. *Let G be a pseudoforest. Then a strong equilibrium can be computed in polynomial time.*

6 Extensions and future work

A natural generalization of our games are coordination games on *weighted* graphs. For these games, Theorem 7 continues to hold, while Theorem 1 does not. An interesting direction which we leave for future work is to derive a characterization of graph classes that guarantee the existence of strong equilibria. Yet another generalization is to allow players to choose multiple colours. Our results on the existence and inefficiency of equilibria then continue to hold; details are deferred to the full version of the paper.

An intriguing open question is whether k-equilibria exist for $k = 3, 4$. Recall that they are guaranteed to exist for $k \leq 2$ and may not exist for $k \geq 5$. Also it would be interesting to derive existence results for other graph classes.

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References

1. N. Andelman, M. Feldman, and Y. Mansour. Strong price of anarchy. *Games and Economic Behavior*, 65(2):289–317, 2009.
2. R. J. Aumann. Acceptable points in general cooperative n-person games. In R. D. Luce and A. W. Tucker, editors, *Contribution to the theory of game IV, Annals of Mathematical Study 40*, pages 287–324. University Press, 1959.
3. H. Aziz and F. Brandl. Existence of stability in hedonic coalition formation games. In *Proc. 11th International Conference on Autonomous Agents and Multiagent Systems*, pages 763–770, 2012.
4. S. Banerjee, Konishi. Core in a simple coalition formation game. *Social Choice and Welfare*, 18(1):135–153, 2001.
5. V. Bilò, A. Fanelli, M. Flammini, and L. Moscardelli. Graphical congestion games. *Algorithmica*, 61(2):274–297, 2011.
6. A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.
7. Cechlárová. Stable partition problem. *Encyclopedia of Algorithms*, pages 885–888, 2008.
8. M. Gairing and R. Savani. Computing stable outcomes in hedonic games. In *Proc. 3rd International Symposium on Algorithmic Game Theory*, pages 174–185, 2010.
9. M. Hoefer. Cost sharing and clustering under distributed competition, 2007. Ph.D. Thesis, University of Konstanz.
10. E. Janovskaya. Equilibrium points in polymatrix games. *Litovskii Matematicheskii Sbornik*, 8:381–384, 1968.
11. D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behaviour*, 14:124–143, 1996.
12. O. Rozenfeld and M. Tennenholtz. Strong and correlated strong equilibria in monotone congestion games. In *Proc. 2nd International Workshop on Internet and Network Economics*, pages 74–86, 2006.