Information Theory Exercise Sheet #7

University of Amsterdam, Master of Logic, Spring 2014
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Out: Thu, 20 March 2014
Due: Fri, 28 March 2014, 9:00

To be solved in Class

1. *Symmetric Channels.* Consider the channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}.$$

In a transition matrix, the entry in the xth row and yth column denotes the conditional probability $P_{Y|X}(y|x)$ that y is received when x has been sent.

Definition 1 *A channel is said to be* symmetric *if the rows of the channel transition matrix $P_{Y|X}$ are permutations of each other and the columns are permutations of each other. A channel is said to be* weakly symmetric *if every row of the transition matrix is a permutation of every other row and all the column sums $\sum_x P_{Y|X}(y|x)$ are equal.*

For instance, the channel $P_{Y|X}$ above is symmetric and the channel

$$Q_{Y|X} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

is weakly symmetric but not symmetric.

(a) Find the optimal input distribution and channel capacity of $Q_{Y|X}$.

(b) Give a general strategy how to compute the capacity for weakly symmetric channels. What is the optimal input distribution?

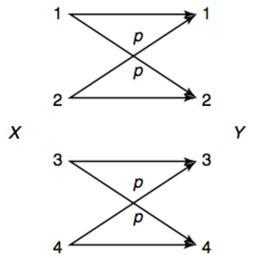
2. *Infinite entropy.* Find a discrete probability distribution $P_X(n)$ with infinite entropy $H(X) = \infty$.
Hint: Recall the integral test for convergence from calculus, saying that the infinite sum $\sum_{n=N}^{\infty} f(n)$ converges to a real number if and only if the integral $\int_N^{\infty} f(x)dx$ is finite. Use the integral test to show that the infinite sum $\sum_{n=2}^{\infty} \frac{1}{n \log(n)}$ diverges, whereas $\sum_{n=2}^{\infty} \frac{1}{n \log^2(n)}$ converges.
Another Hint: Differentiate the functions $f(x) = \log(\log(x))$ and $g(x) = \frac{1}{\log(x)}$.

3. Compute $\frac{d}{dp}h(p)$.

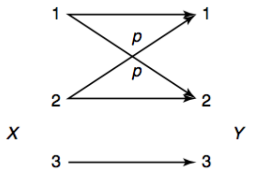
Homework

1. *Capacities.* Find the capacity and optimal input distribution of the following channels:

   (a) [3 points] Two parallel BSCs:



   (b) [6 points] BSC and a single symbol:

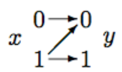


   (c) [3 points] The ternary channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

   (d) [6 points] The Z-channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$



2. Let the input to a channel be a word of 8 bits. The output is also a word of 8 bits. Each time it is used, the channel flips exactly one of the transmitted bits, but the receiver does not know which one. The other seven bits are received without error. All 8 bits are equally likely to be the one that is flipped.

   (a) [2 points] Determine the capacity of this channel.

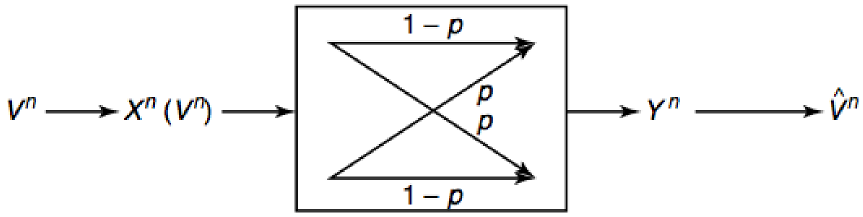
   (b) [4 points] Show, by describing an *explicit encoder and decoder* that the it is possible to communicate with *zero error* 5 bits per channel use.
   Hint: Extend the (7,4) Hamming code to a (8,5) code that does the job.

3. *Noise alphabets.* Consider an additive-noise channel where $\mathcal{X} = \{0, 1, 2, 3\}$ and $Y = X + Z$ with addition over the integers $\mathbb{Z}$. Z is uniformly distributed over three distinct integer values $\mathcal{Z} = \{z_1, z_2, z_3\}$.

   (a) [2 points] What is the maximum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

   (b) [4 points] What is the minimum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

4. [2 points] *Source and channel.* We wish to encode a sequence $V_1, V_2, \ldots$ of iid variables with distribution $P_V(0) = \alpha$, $P_V(1) = 1 - \alpha$ for transmission over a binary symmetric channel with crossover probability p.



Find conditions on α and p so that the probability of error $\Pr[\hat{V}^n \neq V^n]$ can be made to go to zero as $n \to \infty$.