Kolmogorov Complexity, revisited
On Minimum Description Length, Inductive Inference and Machine Learning

Jesus Rodriguez Perez

Universiteit van Amsterdam

December 16, 2014

Outline

The problem of the 'priors'

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$.

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?
  - An hypothesis for the data d is h_1: $d_n := 2n - 1$

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?
  - An hypothesis for the data d is h_1: $d_n := 2n - 1$
  - Another hypothesis is h_2: $d_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?
  - An hypothesis for the data d is h_1: $d_n := 2n - 1$
  - Another hypothesis is h_2: $d_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$
  - Which one of h_1 and h_2 "seems" more probable given the data?.

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?
  - An hypothesis for the data d is h_1: $d_n := 2n - 1$
  - Another hypothesis is h_2: $d_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$
  - Which one of h_1 and h_2 "seems" more probable given the data?.
  - Solution: Pick the hypothesis with highest posterior probability

The problem of the 'priors'

- Consider a computer program looping and printing a number at each iteration. Imagine you pause at a given time the program execution and displayed is the sequence $d := 1, 3, 5, 7$. What number will the computer display in the next iteration?and in the n-th iteration?
  - An hypothesis for the data d is h_1: $d_n := 2n - 1$
  - Another hypothesis is h_2: $d_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$
  - Which one of h_1 and h_2 "seems" more probable given the data?.
  - Solution: Pick the hypothesis with highest posterior probability
  - But how to take a decision with no information other than $\sum_{h_i} h_i = 1$?

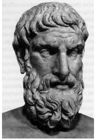
The problem of the 'priors'

The problem of the 'priors'

- Multiple heuristics proposed over the centuries.

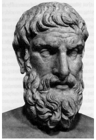
The problem of the 'priors'

- Multiple heuristics proposed over the centuries.
    - *"If more than one theory is consistent with the data, keep them all"*.- Epicurus of Samos (ca. 342 - 270 BC)



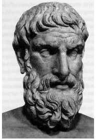
The problem of the 'priors'

- Multiple heuristics proposed over the centuries.
  - *"If more than one theory is consistent with the data, keep them all"*.- Epicurus of Samos (ca. 342 - 270 BC)
  - *"Among competing hypotheses, the simplest one should be selected"*.- William of Ockham (ca. 1287 – 1347)



The problem of the 'priors'

- Multiple heuristics proposed over the centuries.
  - *"If more than one theory is consistent with the data, keep them all"*.- Epicurus of Samos (ca. 342 - 270 BC)

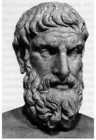


  - *"Among competing hypotheses, the simplest one should be selected"*.- William of Ockham (ca. 1287 – 1347)

- By Occam's Razor, the "simplest" hypothesis is most probable.

The problem of the 'priors'

- Multiple heuristics proposed over the centuries.
  - *"If more than one theory is consistent with the data, keep them all"*.- Epicurus of Samos (ca. 342 - 270 BC)



  - *"Among competing hypotheses, the simplest one should be selected"*.- William of Ockham (ca. 1287 – 1347)

- By Occam's Razor, the "simplest" hypothesis is most probable.But how to define "simple"?

Minimum Description Length

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.
- We may call C an "hypothesis" for d.

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.
- We may call C an "hypothesis" for d.
  - Some C might be *shorter* than the data, so *compress* the data.

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.
- We may call C an "hypothesis" for d.
  - Some C might be *shorter* than the data, so *compress* the data.
- Now consider the following principle: "the best hypothesis for a given set of data is the one that leads to the best compression of the data"

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.
- We may call C an "hypothesis" for d.
  - Some C might be *shorter* than the data, so *compress* the data.
- Now consider the following principle: "the best hypothesis for a given set of data is the one that leads to the best compression of the data"
  - Named Minimum description length principle (due to Jorma Rissanen).

Minimum Description Length

- Consider an encoding of d as a sequence of computer instructions generating d, C=i_1 i_2 ...i_n.
- We may call C an "hypothesis" for d.
  - Some C might be *shorter* than the data, so *compress* the data.
- Now consider the following principle: "the best hypothesis for a given set of data is the one that leads to the best compression of the data"
  - Named Minimum description length principle (due to Jorma Rissanen).
- But this is an instance of Occam's Razor, in which we define "simplest" as "shortest".

Minimum Description Length

Minimum Description Length

- Now we can use it for the initial example:

Minimum Description Length

- Now we can use it for the initial example:

  Recall: x:= 1, 3, 5, 7
  h_1: $x_n := 2n - 1$, h_2: $x_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$.

Minimum Description Length

- Now we can use it for the initial example:

  Recall: x:= 1, 3, 5, 7
  h_1: $x_n := 2n - 1$, h_2: $x_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$.

  want to pick $\hat{h} = \max_{h_1,h_2}\{\frac{P_{D|H}(d|h_1)P_H(h_1)}{P_D(d)}, \frac{P_{D|H}(d|h_2)P_H(h_2)}{P_D(d)}\}$

Minimum Description Length

- Now we can use it for the initial example:

  Recall: x:= 1, 3, 5, 7
  h_1: $x_n := 2n - 1$, h_2: $x_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$.

  want to pick $\hat{h} = \max_{h_1, h_2}\{\frac{P_{D|H}(d|h_1)P_H(h_1)}{P_D(d)}, \frac{P_{D|H}(d|h_2)P_H(h_2)}{P_D(d)}\}$

  since p_2 is longer than p_1 (encodes 9 more arithmetic operations) we have we have $l(p_1) < l(p_2)$ and therefore $P(h_1) > P(h_2)$.

Minimum Description Length

- Now we can use it for the initial example:

  Recall: x:= 1, 3, 5, 7
  h_1: $x_n := 2n - 1$, h_2: $x_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$.

  want to pick $\hat{h} = \max_{h_1, h_2}\{\frac{P_{D|H}(d|h_1)P_H(h_1)}{P_D(d)}, \frac{P_{D|H}(d|h_2)P_H(h_2)}{P_D(d)}\}$

  since p_2 is longer than p_1 (encodes 9 more arithmetic operations) we have we have $l(p_1) < l(p_2)$ and therefore $P(h_1) > P(h_2)$.

  Both h_1 and h_2 are equally consistent with the data, so $P_H(d|h_1) = P_H(d|h_2)$.

  Since $P_D(d)$ is constant we prefer h_1.

Minimum Description Length

- Now we can use it for the initial example:

  Recall: x:= 1, 3, 5, 7
  h_1: $x_n := 2n - 1$, h_2: $x_n := 2n - 1 + (n-1)(n-2)(n-3)(n-4)$.

  want to pick $\hat{h} = \max_{h_1,h_2}\{\frac{P_{D|H}(d|h_1)P_H(h_1)}{P_D(d)}, \frac{P_{D|H}(d|h_2)P_H(h_2)}{P_D(d)}\}$

  since p_2 is longer than p_1 (encodes 9 more arithmetic operations) we have we have $l(p_1) < l(p_2)$ and therefore $P(h_1) > P(h_2)$.

  Both h_1 and h_2 are equally consistent with the data, so $P_H(d|h_1) = P_H(d|h_2)$.

  Since $P_D(d)$ is constant we prefer h_1.

Kolmogorov Complexity

Kolmogorov Complexity

- The length of the shortest code for d is the *Kolmogorov Complexity* of *d*

Kolmogorov Complexity

- The length of the shortest code for d is the *Kolmogorov Complexity* of d
- But the Kolmogorox complexity of any data depends on the language of the program that generates it, so cannot be "universal"...

Kolmogorov Complexity

- The length of the shortest code for d is the *Kolmogorov Complexity* of *d*
- But the Kolmogorox complexity of any data depends on the language of the program that generates it, so cannot be "universal"...

Kolmogorov Complexity

Kolmogorov Complexity

- Good news is that the error we can commit is bounded by the length of the data itself (Invariance Theorem).

Kolmogorov Complexity

- Good news is that the error we can commit is bounded by the length of the data itself (Invariance Theorem).
- Proof:

Kolmogorov Complexity

- Good news is that the error we can commit is bounded by the length of the data itself (Invariance Theorem).
- Proof:Consider $K_p(o) = \min_{p:U(p)=o}\{l(p)\}$ and $K_{p'}(o) = \min_{p':U(p')=o}\{l(p')\}$ for arbitrary p, p' with $|K_p(o) - K_{p'}(o)| = d \geq 1$

Kolmogorov Complexity

- Good news is that the error we can commit is bounded by the length of the data itself (Invariance Theorem).
- Proof:Consider $K_p(o) = \min_{p:U(p)=o} \{l(p)\}$ and $K_{p'}(o) = \min_{p':U(p')=o} \{l(p')\}$ for arbitrary p, p' with $|K_p(o) - K_{p'}(o)| = d \geq 1$

  $d \geq 1 \Rightarrow |\frac{K_p(o)}{m} - \frac{K_{p'}(o)}{m}| < 1$ for m=max$(K_p(o), K_{p'}(o))$.

Kolmogorov Complexity

- Good news is that the error we can commit is bounded by the length of the data itself (Invariance Theorem).
- Proof:Consider $K_p(o) = \min_{p:U(p)=o}\{l(p)\}$ and $K_{p'}(o) = \min_{p':U(p')=o}\{l(p')\}$ for arbitrary p, p' with $|K_p(o) - K_{p'}(o)| = d \geq 1$

  $d \geq 1 \Rightarrow |\frac{K_p(o)}{m} - \frac{K_{p'}(o)}{m}| < 1$ for m=max$(K_p(o), K_{p'}(o))$.

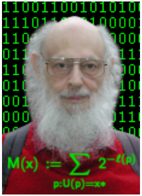
  By definition $K_p(o), K_{p'}(o)) < l(o)$ so that m<l(o)

Solomonoff's inference and Machine Learning

Solomonoff's inference and Machine Learning

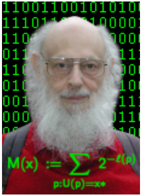
- Kolmogorov Complexity is central in Ray Solomonoff's Inductive Inference Theory.

Solomonoff's inference and Machine Learning



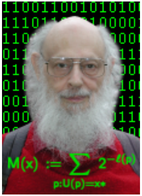
- Kolmogorov Complexity is central in Ray Solomonoff's Inductive Inference Theory.
- The theory formalizes the sequence prediction procedure we did at the beginning of the talk.

Solomonoff's inference and Machine Learning



- Kolmogorov Complexity is central in Ray Solomonoff's Inductive Inference Theory.
- The theory formalizes the sequence prediction procedure we did at the beginning of the talk.
- But sequence prediction is quite a small subset of real-world prediction problems...

Solomonoff's inference and Machine Learning



- Kolmogorov Complexity is central in Ray Solomonoff's Inductive Inference Theory.
- The theory formalizes the sequence prediction procedure we did at the beginning of the talk.
- But sequence prediction is quite a small subset of real-world prediction problems...
- Nevertheless, some Machine Learning problems can be reduced to it.

Solomonoff's inference and Machine Learning

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$
    - Recall first example: Find next in 1,1,2,3,3,5,4,7,5

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$
    - Recall first example: Find next in 1,1,2,3,3,5,4,7,5
- But way to prefer such weird "representation"?

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$
    - Recall first example: Find next in 1,1,2,3,3,5,4,7,5
- But way to prefer such weird "representation"?
  - Suppose some scattered outliers in a dataset.

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$
    - Recall first example: Find next in 1,1,2,3,3,5,4,7,5
- But way to prefer such weird "representation"?
  - Suppose some scattered outliers in a dataset.
  - Traditional ML techniques typically risk fitting the regression function too much (high P(h—d)) with complex models (low P(h) as estimated by Occam's Razor).

Solomonoff's inference and Machine Learning

- Discrete regression is a good example:
  - Training dataset (feature,value): $(x_1, f(x_1)), (x_2, f(x_2)), ...(x_n, f(x_n))$.
  - Goal: Find f
  - Equivalent to find $f(i) = f(x_{n+1})$ given $x_1 f(x_1) x_2 f(x_2) x_n f(x_n) x_{n+1}$
    - Recall first example: Find next in 1,1,2,3,3,5,4,7,5
- But way to prefer such weird "representation"?
  - Suppose some scattered outliers in a dataset.
  - Traditional ML techniques typically risk fitting the regression function too much (high P(h—d)) with complex models (low P(h) as estimated by Occam's Razor).
  - In contrast, MDL principle tends to gain a balance with $P(d|h) \approx P(h)$, therefore maximizing $P(d|h)P(h)$.

Conclusions

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.
- MDL pple performs better than typical ML models in very noisy data.

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.
- MDL pple performs better than typical ML models in very noisy data.
- Unlike entropy, kolmogorov complexity cannot be computed in general.

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.
- MDL pple performs better than typical ML models in very noisy data.
- Unlike entropy, kolmogorov complexity cannot be computed in general.
  - However Kolmogorov Complexity is "approximately" computable.

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.
- MDL pple performs better than typical ML models in very noisy data.
- Unlike entropy, kolmogorov complexity cannot be computed in general.
  - However Kolmogorov Complexity is "approximately" computable. We can avoid infinite loops at the cost of an approximated solution:

Conclusions

- Sometimes little training data is available, and MDL pple assumes nothing about the distributions it tries to compress, so has a general advantage to Maximum Entropy models, which assume a uniform distribution of a restricted set of parameters.
- MDL pple performs better than typical ML models in very noisy data.
- Unlike entropy, kolmogorov complexity cannot be computed in general.
  - However Kolmogorov Complexity is "approximately" computable. We can avoid infinite loops at the cost of an approximated solution:
    - e.g. Time-bounded "Levin" complexity: $\hat{K}(o) := \min_{p:U(p)=o \text{ in } t \text{ steps}} \{l(p) + logt\}$

Conclusions