Information Theory Exercise Sheet #7

(Channel Capacities)

University of Amsterdam, Master of Logic, Fall 2015
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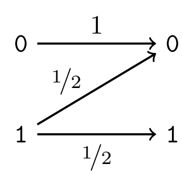
Out: Wednesday, 9 December 2015
(due: Friday, 18 December 2015, 9:00)

1. **Channel Capacities (Cover and Thomas, Ex. 7.34)** Find the capacity and optimal input distribution of the following channels:

   (a) The Z-channel with transition matrix

   $$P_{Y|X} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

   and channel diagram

   (b) The ternary channel with transition matrix

   $$P_{Y|X} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

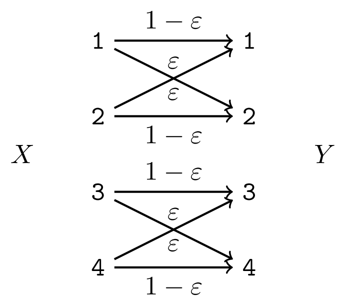
2. **Toggle channel (Cover and Thomas, Ex. 7.28)** Suppose we are given two (possibly different) channels with capacities C_1 and C_2. We then define a new "union channel" by allowing the transmitter at each time to choose between sending a signal through either channel 1 or channel 2. We assume that that the output alphabets of the two component channels are distinct and do not intersect.

   (a) Letting C stand for the capacity of the union channel, show that

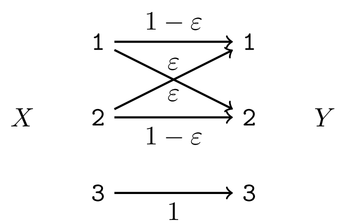
   $$2^C = 2^{C_1} + 2^{C_2}.$$

   Use this result to calculate the capacity of the following channels:

(b) Two parallel BSCs:



(c) A BSC and a single symbol:



3. **Noise alphabets (Cover and Thomas, Ex. 7.30)** Consider an additive-noise channel where $\mathcal{X} = \{0, 1, 2, 3\}$ and $Y = X + Z$ with addition over the integers $\mathbb{Z}$. Z is uniformly distributed over three distinct integer values $\mathcal{Z} = \{z_1, z_2, z_3\}$.

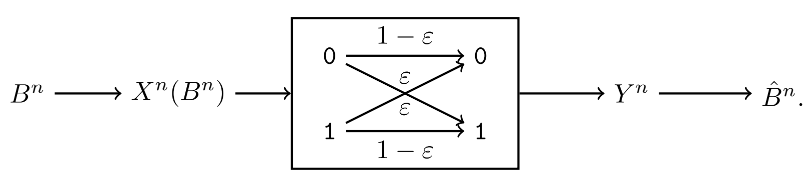
   (a) What is the maximum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

   (b) What is the minimum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

4. **Source and channel (Cover and Thomas, Ex. 7.31)** We wish to encode a sequence $B_1, B_2, B_3, \ldots, B_n$ of i.i.d. Bernoulli(θ) variables,

$$P_B(1) \;=\; 1 - P_B(0) \;=\; \theta,$$

   for transmission over a binary symmetric channel with flip probability ε,



   Find conditions on (θ, ε) under which the error probability $\Pr[\hat{B}^n \neq B^n]$ can be made to go to zero as $n \to \infty$.

5. **Another Kind of Entropy.** In this exercise we consider a different entropy notion. Let X and Y be random variables with joint probability distribution P_{XY}. The *guessing probability* and the *min-entropy* of X are respectively defined as

$$\mathrm{Guess}(X) := \max_x P_X(x) \quad \text{and} \quad H_{\min}(X) := -\log \mathrm{Guess}(X).$$

The *conditional guessing probability* and the *conditional min-entropy* of X are respectively defined as

$$\mathrm{Guess}(X|Y) := \sum_y P_Y(y)\mathrm{Guess}(X|Y = y)$$

and

$$H_{\min}(X|Y) := -\log \mathrm{Guess}(X|Y).$$

(a) If X has no uncertainty (i.e. $H(X) = 0$), what is $H_{\min}(X)$?

(b) If X is uniformly distributed over $\mathcal{X}$, what is $H_{\min}(X)$?

(c) Prove that $H_{\min}(XY) \geq H_{\min}(X)$.

(d) Prove that $H_{\min}(X) \geq H_{\min}(X|Y)$.

(e) Prove that $H_{\min}(X|Y) \geq H_{\min}(XY) - \log |\mathcal{Y}|$.

Homework is exercises 1, 2, 3, and 4.