Selected Areas in Cryptology
Cryptanalysis
Week 3

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Block cipher structural attacks

Attacks against the internal structure of a blockcipher

$E_K : \{0,1\}^n \to \{0,1\}^n, \quad K \in \{0,1\}^k$

Blockcipher consists of R rounds of a small keyed round function E_K^r

- Small: few operations
- Keyed: involves key material
- 'Confusion': complex operations ⇒ very complex final relations
- 'Diffusion': mix state ⇒ each in-/output bit depends on each out-/input bit

Focus on SPN: Substitution Permutation Network

- Substitution: complex permutation "S-BOX" on e.g. 8 bits applied on all 8-bit parts
- Permutation: mixing of entire state (F_2 - linear)
- Keyed: add round key (F_2 - linear) (derived from main key)

AES: state $n = 128$ bits, key $k = 128{,}192{,}256$ bits, S-box: 8 bits

ToyCipher: state $n = 16$ bits, key $k = 16(r+1)$ bits, S-box: 4 bits

Toy-Cipher

Toy-Cipher to demonstrate structural attack techniques

- State $n = 16$ bits, 4 rounds
- 5 round keys $K_1, \dots, K_5$ of 16 bits
- Small enough to do attacks in practice (if you wanted)

Key-addition:

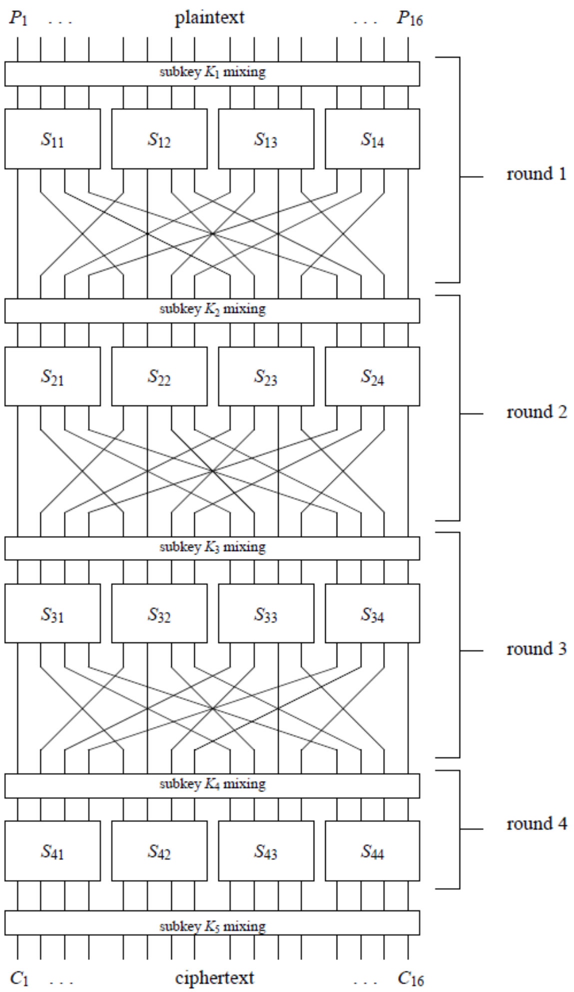
- XOR round key K_i
- Final key-addition at end with K_5

Substitution: 4-bit S-box

- $\pi_S: \{0,1\}^4 \to \{0,1\}^4$ (see lecture notes)
- called 4 times per round to alter all 16 bits

Permutation of 16 bits:

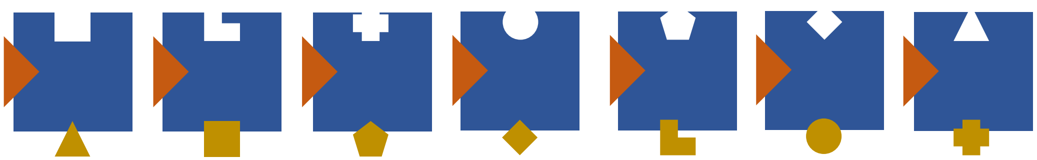
- $\pi_P: \{1, \dots, 16\} \to \{1, \dots, 16\}$ (see lecture notes)
- Skipped in last round, as it can be removed anyway (swap Perm and AddKey with $K_5'[i] = K_5[\pi_P(i)]$)



Structural attacks

1. Analyze individual rounds

2. Obtain a family of round attack building blocks

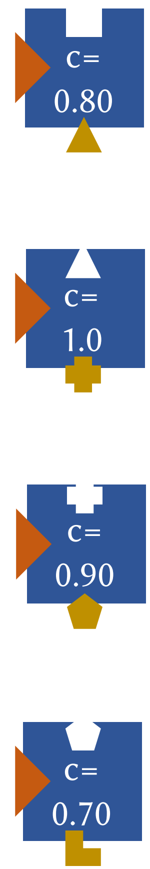


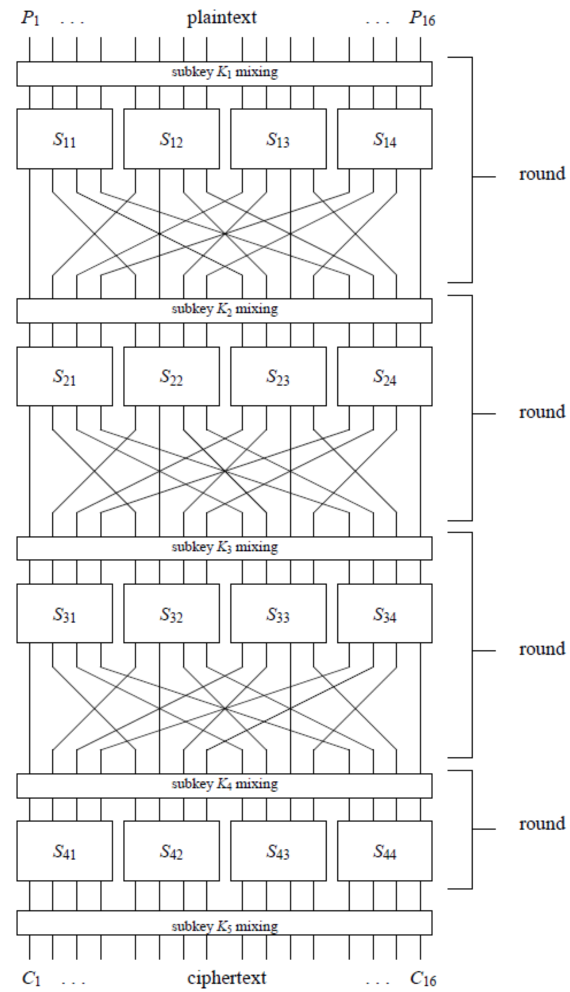
3. Combine to attack on full blockcipher

4. Approximate complexity
by combining individual round costs

$$C = c(r) \cdot 0.8 \cdot 1.0 \cdot 0.9 \cdot 0.7$$

5. Find optimal attack





Linear Cryptanalysis

Linear approximate the cipher:

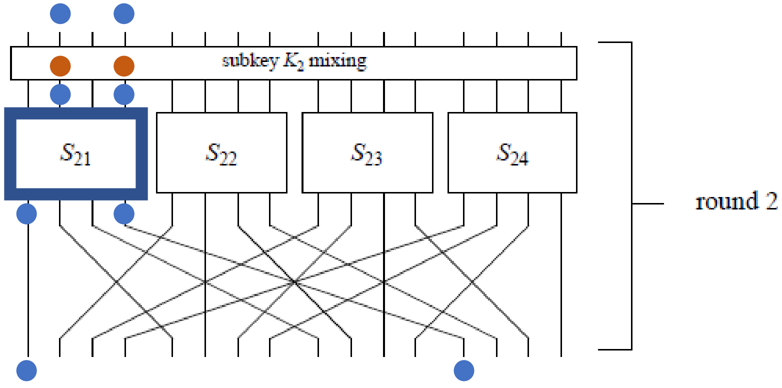
- F_2- linear input-output relation
- $\sum_{i \in I} P[i] \quad \oplus \quad \sum_{j \in J} C[j] \quad \oplus \sum_{l \in L} K[l] = c$
  - Involves a number of plaintext bits $P[i]$,
  - .. cipher text bits $C[j]$,
  - .. key bits $K[l]$, (from all the round keys)
  - .. a constant c
- E.g.: $P[2] \oplus P[4] \oplus C[1] \oplus C[7] \oplus K_1[2] \oplus K_1[4] \cdots \oplus K_5[7] = 1$

- F_2: Either the equation holds with $c = 0$ or with $c = 1$
- Probability equation holds:
  - Ideal secure situation: $p = 0.5$ exactly for any such relation
  - $\Rightarrow$ approximation doesn't give any information
  - Actual case $p = 0.5 + \epsilon$, where $\epsilon \in [-.5, +.5]$ is the bias
  - $\Rightarrow$ larger bias means larger probability of correct prediction
- Search for relations with large (absolute) bias!

- First find relations on individual rounds, then combine them!

Linear Cryptanalysis

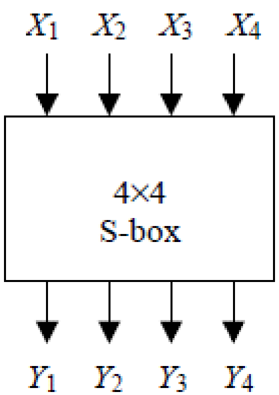
- A forward analysis
  - Round input: $P[1], \dots, P[16]$
  - S-Box input: $X[1], \dots, X[16]$
  - S-Box output: $Y[1], \dots, Y[16]$
  - Round output: $C[1], \dots, C[16]$
  - Round key: $K[1], \dots, K[16]$
- Choose input bits: $P[2], P[4]$
- Involves key bits $K[2], K[4]$
- Inactive S-Box: no input bits selected
- 1 active S-Box: S_{21}
  - Inputs:
    - $X[2] = P[2] \oplus K[2]$
    - $X[4] = P[4] \oplus K[4]$
  - Choose outputs: $Y[1], Y[4]$
- Resulting output bits: $C[1] = Y[1], C[13] = Y[4]$



- Relation: Rel: $P[2] \oplus P[4] \oplus C[1] \oplus C[13] = 0 \oplus K[2] \oplus K[4]$
- Probability:
  - $\Pr_P[Rel] = \Pr_X[X[2] \oplus X[4] \oplus Y[1] \oplus Y[4] = 0 \mid Y = \pi_S(X)]$

LAT: Linear Approximation Table

- Analyze all linear relations for S-Box π_S of the form:
  - $\Pr_X[X[2] \oplus X[4] \oplus Y[1] \oplus Y[4] = 0 \mid Y = \pi_S(X)]$



- S-Box is permutation on $\{0,1\}^4$
  - 16 possible selections of sums $\sum_{i \in I} X[i]$, $I \subseteq \{1,2,3,4\}$
  - 16 possible selections of sums $\sum_{j \in I} Y[j]$, $J \subseteq \{1,2,3,4\}$
  - Represent I/J as 4-bit mask / integer value:

    $\{1\} \to 1000_b = 8, \qquad \{3,4\} \to 0011_b = 3$

- Linear Approximation Table (LAT):
  - 16 x 16 table
  - Row $I \in \{0, \ldots, 15\}$, Column $J \in \{0, \ldots, 15\}$ contains:
  - $LAT(I,J) := \#\{X \in \{0,1\}^4, Y = \pi_S(X) \mid \sum X[i] \oplus \sum Y[j] = 0\} - 8$
  - Bias $\epsilon_{I,J} = \Pr[\sum X[i] \oplus \sum Y[j] = 0] - 0.5 = LAT(I,J)/16$
  - Important tool!
    - Easily precomputed, independent of keys
    - Convenient look-up for large biases to construct large bias relations

LAT: Linear Approximation Table

- Compute entry
  1. Write all values for X with corresponding Y-values
  2. Compute X-sum
  3. Compute Y-sum
  4. Count total matching values ($A \oplus B = 0 \Leftrightarrow A = B$)
  5. Subtract 8

$X_1X_2X_3X_4$	$Y_1Y_2Y_3Y_4$	$X_2 + X_3$	$Y_1 + Y_3 + Y_4$
0000	1110	0	0
0001	0100	0	0
0010	1101	1	0
0011	0001	1	1
0100	0010	1	1
0101	1111	1	1
0110	1011	0	1
0111	1000	0	1
1000	0011	0	0
1001	1010	0	0
1010	0110	1	1
1011	1100	1	1
1100	0101	1	1
1101	1001	1	0
1110	0000	0	0
1111	0111	0	0

- $X[2] \oplus \mathrm{X}[3] \oplus Y[1] \oplus Y[3] \oplus Y[4]$:
  - 12 matching
  - $\Pr[\sum = 0] = \frac{12}{16}$, $\epsilon = \frac{12}{16} - \frac{1}{2} = \frac{4}{16}$
  - $x = 0110_b = 6$
  - $y = 1011_b = 11$
  - $\Rightarrow LAT(6,11) = 12 - 8 = 4$

LAT: Linear Approximation Table

LAT for Toy-Cipher

LAT properties:

- $LAT(0,0) = 16 - 8 = 8,\ \ LAT(x,0) = 8 - 8,\ \ LAT(0,x) = 8 - 8,\ \ x > 0$

Also note:

Every entry is even

Sum of every row/column = 8

Input sum	Output sum															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-2	-2	0	0	-2	6	2	2	0	0	2	2	0	0
2	0	0	-2	-2	0	0	-2	-2	0	0	2	2	0	0	-6	2
3	0	0	0	0	0	0	0	0	2	-6	-2	-2	2	2	-2	-2
4	0	2	0	-2	-2	-4	-2	0	0	-2	0	2	2	-4	2	0
5	0	-2	-2	0	-2	0	4	2	-2	0	-4	2	0	-2	-2	0
6	0	2	-2	4	2	0	0	2	0	-2	2	4	-2	0	0	-2
7	0	-2	0	2	2	-4	2	0	-2	0	2	0	4	2	0	2
8	0	0	0	0	0	0	0	0	-2	2	2	-2	2	-2	-2	-6
9	0	0	-2	-2	0	0	-2	-2	-4	0	-2	2	0	4	2	-2
10	0	4	-2	2	-4	0	2	-2	2	2	0	0	2	2	0	0
11	0	4	0	-4	4	0	4	0	0	0	0	0	0	0	0	0
12	0	-2	4	-2	-2	0	2	0	2	0	2	4	0	2	0	-2
13	0	2	2	0	-2	4	0	2	-4	-2	2	0	2	0	0	2
14	0	2	2	0	-2	-4	0	2	-2	0	0	-2	-4	2	-2	0
15	0	-2	-4	-2	-2	0	2	0	0	-2	4	-2	-2	0	2	0

Compute with sage (see lecture notes)

Piling-Up Lemma

How to combine two linear relations ?

- Let X_1, X_2 be two independent binary random variables
  (think of them as the output of the sum of X & Y bits)
- Let $p_1 := \Pr[X_1 = 0]\,, p_2 := \Pr[X_2 = 0]$
- Then:
  $\Pr[X_1 \oplus X_2 = 0]$
  $= \Pr[X_1 = 0 \wedge X_2 = 0] + \Pr[X_1 = 1 \wedge X_2 = 1]$
  $= p_1 \cdot p_2 + (1 - p_1) \cdot (1 - p_2)$
- Now consider the biases:
  $\epsilon_1 := p_1 - 0.5, \quad \epsilon_2 := p_2 - 0.5, \quad \epsilon_{1,2} := \Pr[X_1 \oplus X_2 = 0] - 0.5$
- Then:
  $\epsilon_{1,2} = (0.5 + \epsilon_1)(0.5 + \epsilon_2) + (0.5 - \epsilon_1)(0.5 - \epsilon_2) - 0.5$
  $= (0.25 + 0.5(\epsilon_1 + \epsilon_2) + \epsilon_1\epsilon_2) + (0.25 - 0.5(\epsilon_1 + \epsilon_2) + \epsilon_1\epsilon_2) - 0.5$
  $= 2\epsilon_1\epsilon_2$

Piling-Up Lemma:

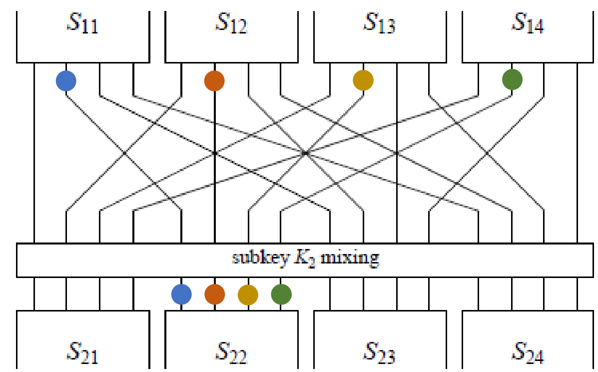
For $X_1, \ldots, X_N$ independent binary variables with biases ϵ_i:

Their sum $X_{1,\ldots,N} = X_1 \oplus \cdots \oplus X_N$ has bias: $\epsilon_{1,\ldots,N} = 2^{N-1} \prod_{i=1}^{N} \epsilon_i$

Bringing everything together

- LAT to find those high bias S-Box relations
- Inactive S-Boxes don't affect bias, as:
  - $LAT(0,0) = 8 \Rightarrow \epsilon_1 = \frac{8}{16} = \frac{1}{2}$
  - Piling-Up Lemma: $\epsilon_{1,2} = 2\epsilon_1\epsilon_2 = \epsilon_2$
- Only active S-Boxes matter ⇒ minimize active S-boxes

- Make use of π_P properties
  - i-th output bit active of S-Box S_{1j} ⇒ S-Box S_{2i} active in next round
  - It is its own inverse, so also vice-versa:
  - i-th input bit active of S-Box S_{2j} ⇒ S-Box S_{1i} active in previous round

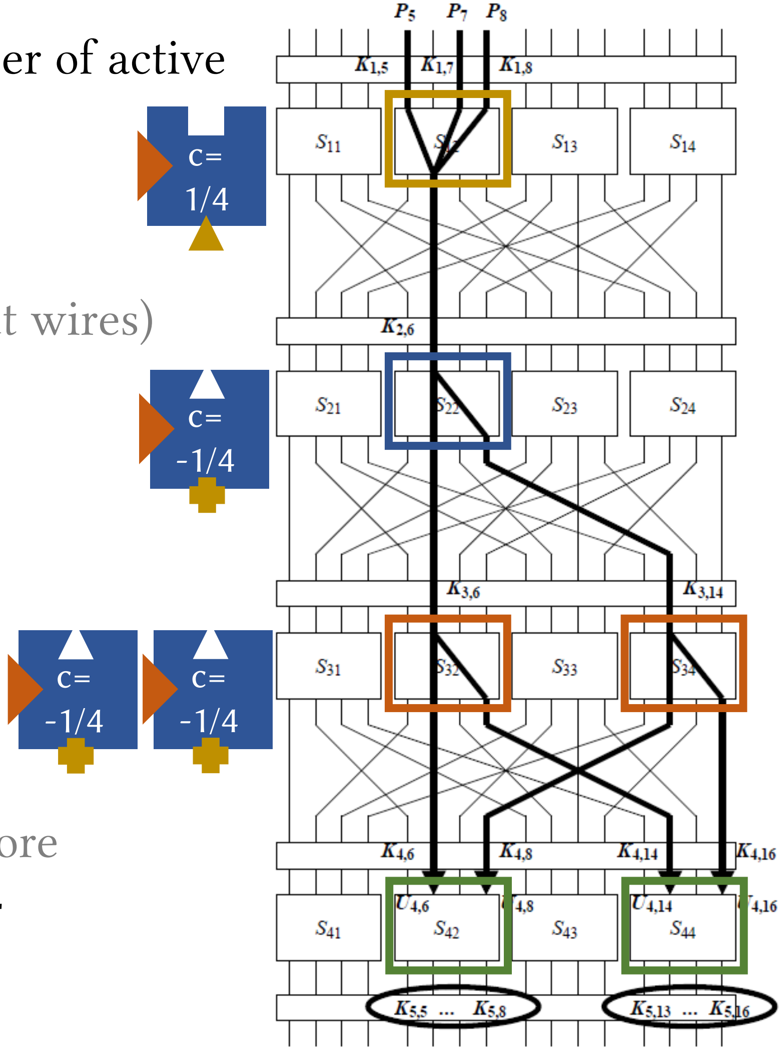


- If multiple active S-boxes in one round then try to have active input bits on same S-box bit position (and same for output bits)

Bringing everything together

Goal is to build a linear approximation over <u>three</u> rounds

- First find S-Box relation for <u>middle round</u> with <u>high bias</u> and <u>minimal active wires</u>
  - The number of active wires equals the number of active S-Boxes in round 1 and 3 together
- E.g.: $\mathrm{LAT}(0100_b, 0101_b) = \mathrm{LAT}(4,5) = -4$
- If we use it at S-Box 2 (0100) then next round:
  - Has 2 active S-Boxes (0101_b: 2 active output wires)
  - Both have active input wire 2 $\Rightarrow 0100_b$
- So can use same high bias relation again
  - $\Rightarrow$ rounds 2 and 3 done
  - Round 4 has 2 active S-Boxes
- First round:
  - Active S-Box 2 with output mask 0100_b
  - Find highest bias
  - Input mask is not important: no S-Boxes before
  - E.g. $\mathrm{LAT}(1011_b, 0100_b) = \mathrm{LAT}(11,4) = 4$



Bringing everything together

First round:

- $X_{12,1} \oplus X_{12,3} \oplus X_{12,4} = P_5 \oplus P_7 \oplus P_8 \oplus K_{1,5} \oplus K_{1,7} \oplus K_{1,8}$
- $X_{12,1} \oplus X_{12,3} \oplus X_{12,4} \oplus Y_{12,2} = 0$ with bias $\epsilon_{12} = 4/16$

Second round:

- $X_{22,2} = Y_{12,2} \oplus K_{2,6}$
- $X_{22,2} \oplus Y_{22,2} \oplus Y_{22,4} = 0$ with bias $\epsilon_{22} = -4/16$

Third round:

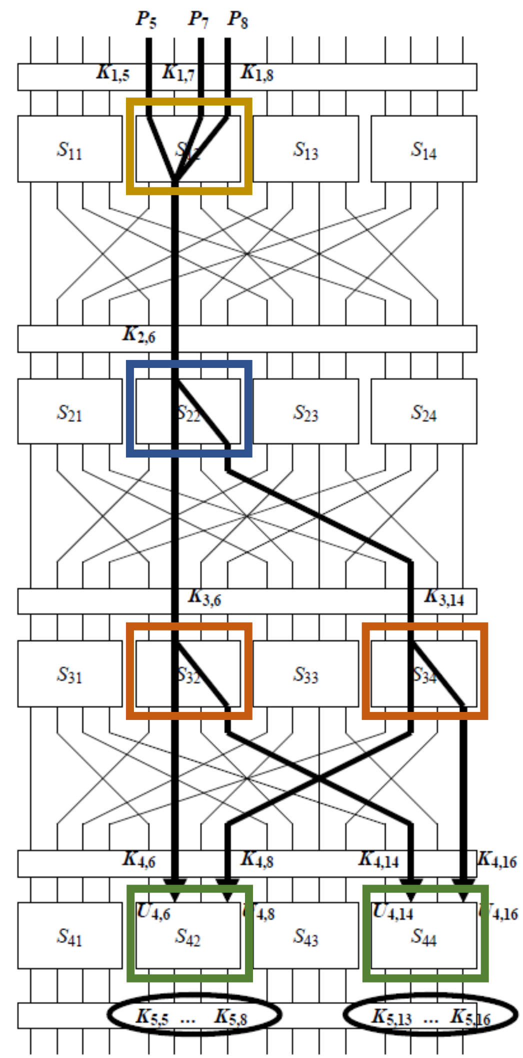
- $X_{32,2} = Y_{22,2} \oplus K_{3,6}, \quad X_{34,2} = Y_{22,4} \oplus K_{3,14}$
- $X_{32,2} \oplus Y_{32,2} \oplus Y_{32,4} = 0$ with bias $\epsilon_{32} = -4/16$
- $X_{34,2} \oplus Y_{34,2} \oplus Y_{34,4} = 0$ with bias $\epsilon_{34} = -4/16$

Partial fourth round:

- $X_{42,2} \oplus X_{42,4} = Y_{32,2} \oplus Y_{34,2} \oplus K_{4,6} \oplus K_{4,8}$
- $X_{44,2} \oplus X_{44,4} = Y_{32,4} \oplus Y_{34,4} \oplus K_{4,14} \oplus K_{4,16}$

Sum all relations above (move only key bits on RSH):

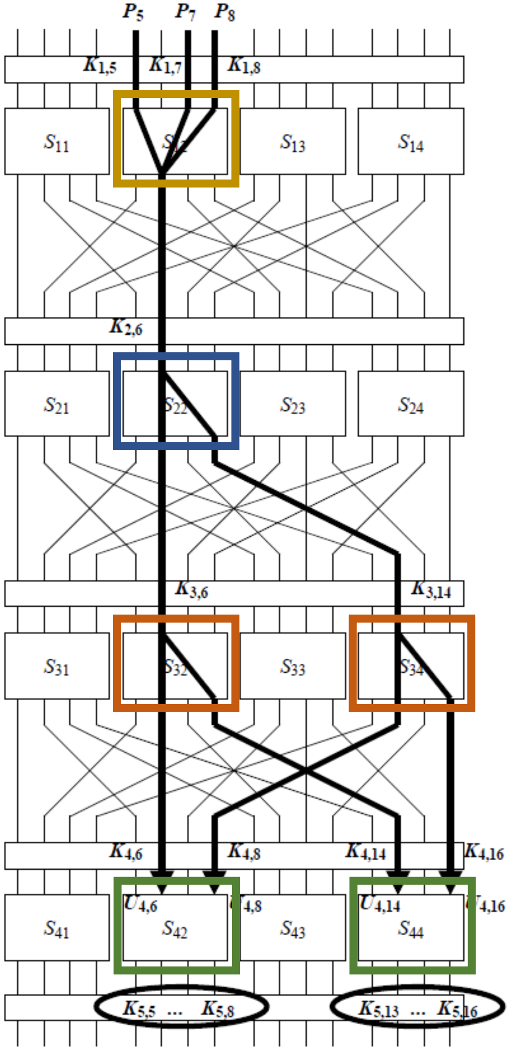
- $P_5 \oplus P_7 \oplus P_8 \oplus X_{42,2} \oplus X_{42,4} \oplus X_{44,2} \oplus X_{44,4} =$
  $K_{1,5} \oplus K_{1,7} \oplus K_{1,8} \oplus K_{2,6} \oplus K_{3,6} \oplus K_{3,14} \oplus K_{4,6} \oplus K_{4,8} \oplus K_{4,14} \oplus K_{4,16}$
- Note how all internal variables occur exactly twice & cancel
- Bias (Piling-Up Lemma): $2^3 \left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)^3 = -\frac{1}{32}$



Key-recovery attack

$P_5 \oplus P_7 \oplus P_8 \oplus X_{42,2} \oplus X_{42,4} \oplus X_{44,2} \oplus X_{44,4} =$
$K_{1,5} \oplus K_{1,7} \oplus K_{1,8} \oplus K_{2,6} \oplus K_{3,6} \oplus K_{3,14} \oplus K_{4,6} \oplus K_{4,8} \oplus K_{4,14} \oplus K_{4,16}$

With bias: $2^3 \left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)^3 = -\frac{1}{32}$



Build distinguisher for 3 rounds (w/ 4 key additions)

- Over many plaintext-ciphertext pairs measure probability of relation
- Is $\approx \pm\frac{1}{32} \Rightarrow$ is blockcipher oracle with 3 rounds
- Is $\approx 0.5 \Rightarrow$ random oracle

Key-recovery attack idea:

1. Obtain many plaintext-ciphertext pairs
2. Guess last round key => decrypt last round
   - Note how we only need to guess 8 key bits of K_5
3. Do distinguishing check
   - Outputs blockcipher oracle
     $\Rightarrow$ right key guess, stop
   - Outputs random oracle
     $\Rightarrow$ wrong key guess, try again with another guess

Key-recovery attack analysis

Count P-C pairs that match relation: C

Case correct key-guess:

- Binomial distribution with n samples and $p = 0.5 + \epsilon$
- $E[C] = n/2 + n \cdot \epsilon$

Case wrong key-guess:

- Binomial distribution with n samples and $p = 0.5$
- $E[C] = n/2$

However, there are $\approx 2^8$ wrong key-guesses

- Does the correct key-guess stand out among <u>all of them?</u>
- Approximate with Normal distribution N: mean $n/2$ and SD $\sqrt{n/4}$
- Then $\Pr[|N - mean| > x \cdot SD] \leq e^{-x^2/2}$ (see lecture notes)
- For $x = 4$, this probability is $\ll 2^{-8} \Rightarrow$ expect all samples bounded by $4 \cdot SD$

How many samples do we need to have the correct key-guess stand out?

- $n \cdot \epsilon > 4 \cdot \sqrt{n/4} \quad \Rightarrow \quad n > 4 \cdot \epsilon^{-2}$

Wrap-up

- Block-cipher design:
  - Substitution: S-Box
  - Permutation: linear
  - Key-addition: linear
- Linear cryptanalysis
  - Input/output- linear relations with probability bias
  - LAT: Linear Approximation Table for S-Box
  - Build linear relation for block cipher
    by combining internal linear relations with piling-up lemma
- Linear distinguisher
  - Blockcipher oracle vs Random oracle
  - Distinguish by measuring non-zero bias vs zero bias
- Key-recovery attack
  - Use distinguisher on R-1 rounds
  - Guess last key and distinguish: random oracle ⇒ wrong key guess
  - Number of P-C pairs: $O(\epsilon^{-2})$