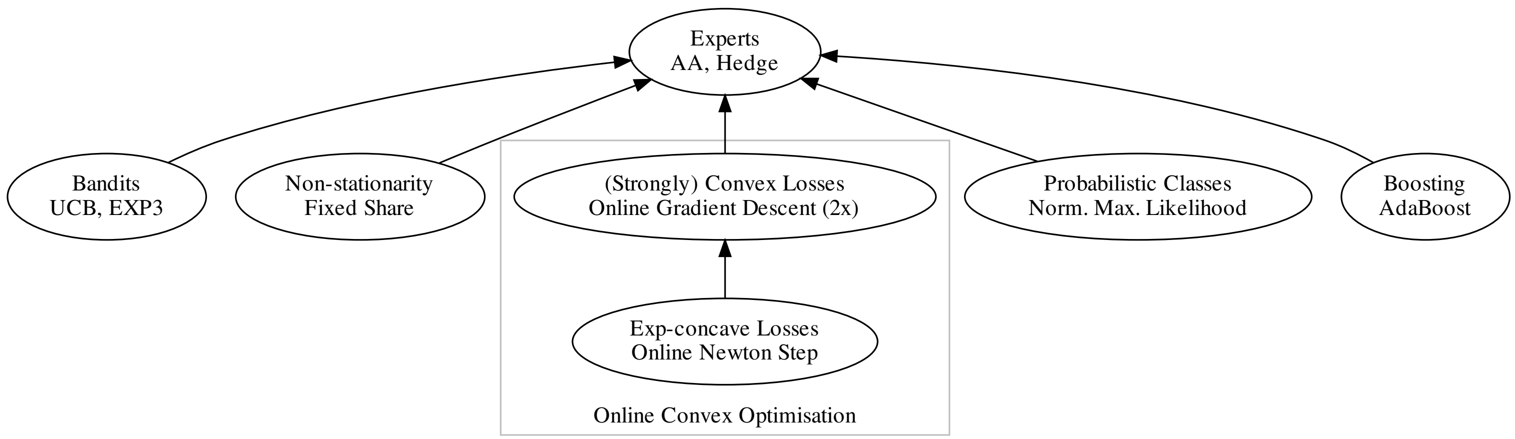
Machine Learning Theory. Lecture 7.

Wouter M. Koolen

- Online Learning, Experts, basic algorithms



Online learning focus

- Tight feedback loop (recurring prediction task)
- Continuous learning (no training/learning separation)
- Adversarial analysis (Prequential principle, individual sequence. There is only the data. Also establishes robustness of statistical estimators.)
- Emphasis on both computational and statistical performance
- Regret: relative notion of performance

Application domains

Truly sequential problems:

- electricity demand prediction (EDF, also Amazon)
- mobile device power management
- hybrid cars engine switching
- caching
- medical trials (bandits)
- online advertisement (bandits)
- weather forecasting
- data compression (CTW)
- statistical testing
- investment (Universal portfolios)
- input assistants (e.g. Dasher)
- prediction with expert advice (meld human and machine prediction)
- online convex optimisation

Wider application

- Big data sets (transport state of online algorithm instead of data, online to batch conversion)
- Convex optimisation
- Game theory (online learning methods for approximate equilibrium)
- General understanding
  - Uncertainty and ways to manipulate it
  - Makeup of and patterns in data
  - Complexity of classes of strategies

The menu for today

Two fundamental and prototypical online learning problems

- The mix-loss game
  - Aggregating Algorithm
- The dot-loss game
  - Hedge Algorithm

Mix-loss game

Protocol:

- For $t = 1, 2, \ldots$
  - Learner chooses a distribution $\boldsymbol{w}_t \in \triangle_K$ on K “experts”.
  - Adversary reveals loss vector $\boldsymbol{\ell}_t \in (-\infty, \infty]^K$.
  - Learner’s loss is the **mix loss** $-\ln\left(\sum_{k=1}^K w_t^k e^{-\ell_t^k}\right)$

Instances:

- Investment (loss is *negative log-growth*)
- Data compression (loss is *code length*)
- Probability forecasting (loss is *logarithmic loss*)

Mix-loss objective

Obviously we cannot guarantee small loss.

Idea: relative evaluation, i.e. seek performance close to best expert.

Definition: After T rounds of the mix-loss game, the *regret* is given by

$$R_T = \underbrace{\sum_{t=1}^{T} -\ln\left(\sum_{k=1}^{K} w_t^k e^{-\ell_t^k}\right)}_{\text{Learner's mix loss}} - \underbrace{\min_k \sum_{t=1}^{T} \ell_t^k}_{\text{loss of best expert}}$$

Goal: design a strategy for Learner that guarantees low regret.

Worst-case regret and Minimax regret

An strategy for the learner assigns a next action $\boldsymbol{w}_t$ to each history $(\boldsymbol{w}_1, \boldsymbol{\ell}_1), \ldots, (\boldsymbol{w}_{t-1}, \boldsymbol{\ell}_{t-1})$.

Definition: The *worst-case regret* of a strategy S for the learner is

$$\max_{\boldsymbol{\ell}_1}, \ldots, \max_{\boldsymbol{\ell}_T} R_T$$

where the $\boldsymbol{w}_t$ are chosen according to S.

Definition: The *minimax regret* of the mix loss game is

$$\min_{\boldsymbol{w}_1} \max_{\boldsymbol{\ell}_1}, \ldots, \min_{\boldsymbol{w}_T} \max_{\boldsymbol{\ell}_T} R_T$$

Mix-loss regret: lower bound (adversary construction)

Theorem: Any strategy for Learner has worst-case regret $\geq \ln K$, *already in $T = 1$ round.*

Idea: Look at $k_{\text{low}} \in \arg\min_k w_1^k$ so that $w_1^{k_{\text{low}}} \leq \frac{1}{K}$.

Administer loss killing everyone but k_{low}

$$\ell_1^k = \begin{cases} \infty & k \neq k_{\text{low}} \\ 0 & k = k_{\text{low}} \end{cases}$$

Now Learner's mix loss equals

$$-\ln\left(\sum_{k=1}^K w_1^k e^{-\ell_1^k}\right) = -\ln\left(w_1^{k_{\text{low}}} e^{-\ell_t^{k_{\text{low}}}}\right) \geq \ln K + \ell_t^{k_{\text{low}}}$$

The Aggregating Algorithm for mix loss

Definition: The *Aggregating Algorithm* plays weights in round t:

$$w_t^k = \frac{e^{-\sum_{s=1}^{t-1} \ell_s^k}}{\sum_{j=1}^K e^{-\sum_{s=1}^{t-1} \ell_s^j}} \qquad \text{(AA)}$$

or, equivalently, $w_1^k = \frac{1}{K}$ and

$$w_{t+1}^k = \frac{w_t^k e^{-\ell_t^k}}{\sum_{j=1}^K w_t^j e^{-\ell_t^j}} \qquad \text{(AA, incremental)}$$

Many names

- (Generalisation of) Bayes' rule
- Exponentially weighted average

Mix-loss regret: upper bound (algorithm)

Theorem: The regret of the Aggregating Algorithm does not exceed $R_T \le \ln K$ for all $T \ge 0$.

Proof: Crucial observation is that mix loss *telescopes*

$$\sum_{t=1}^{T} -\ln\left(\sum_{k=1}^{K} w_t^k e^{-\ell_t^k}\right) = \sum_{t=1}^{T} -\ln\left(\sum_{k=1}^{K} \frac{e^{-\sum_{s=1}^{t-1} \ell_s^k}}{\sum_{j=1}^{K} e^{-\sum_{s=1}^{t-1} \ell_s^j}} e^{-\ell_t^k}\right)$$

$$= \sum_{t=1}^{T} -\ln\left(\frac{\sum_{k=1}^{K} e^{-\sum_{s=1}^{t} \ell_s^k}}{\sum_{j=1}^{K} e^{-\sum_{s=1}^{t-1} \ell_s^j}}\right)$$

$$= -\ln\left(\sum_{k=1}^{K} e^{-\sum_{t=1}^{T} \ell_t^k}\right) + \ln K.$$

Bounding the sum from below by a max results in

$$\leq \min_k \sum_{t=1}^{T} \ell_t^k + \ln K \tag{1}$$

Dot-loss game

Protocol:

- For $t = 1, 2, \ldots$
  - Learner chooses a distribution $\boldsymbol{w}_t \in \triangle_K$ on K "experts".
  - Adversary reveals loss vector $\boldsymbol{\ell}_t \in [0,1]^K$.
  - Learner's loss is the **dot loss** $\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t = \sum_{k=1}^K w_t^k \ell_t^k$

Many names:

- Decision Theoretic Online Learning
- Prediction with Expert Advice
- The Hedge setting

Dot-loss objective

Definition: *Regret* after T rounds:

$$R_T \;=\; \sum_{t=1}^{T} \boldsymbol{w}_t^{\intercal} \boldsymbol{\ell}_t - \min_k \sum_{t=1}^{T} \ell_t^k$$

Goal: design an algorithm for Learner that guarantees low regret.

Mix loss vs Dot-loss (Jensen)

By Jensen's Inequality for the convex function $x \mapsto -\ln(x)$

$$\underbrace{-\ln\left(\sum_{k=1}^{K} w_t^k e^{-\ell_t^k}\right)}_{\text{mix loss}} \leq \underbrace{\sum_{k=1}^{K} w_t^k \ell_t^k}_{\text{dot loss}} \tag{2}$$

So the dot loss game is harsher for the Learner …

… but maybe we can find a converse inequality (with small overhead)

Mix loss vs Dot loss (Hoeffding)

Lemma 1 (Hoeffding)**.** *Fix zero-mean r.v.* $X \in [a, b]$*, and let* $\eta \in \mathbb{R}$*. Then*

$$\mathbb{E}[e^{\eta X}] \;\leq\; e^{\eta^2 (b-a)^2/8}$$

(Note: Lemma is main ingredient of but not equal to Hoeffding's Bound)

Application: Fix $\boldsymbol{w}_t \in \triangle_K$ and $\boldsymbol{\ell}_t \in [0,1]^K$. Define r.v. X to take value $\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t - \ell_t^k$ with probability w_t^k for all $k = 1, \ldots, K$. Then X has mean zero, and takes values in an interval of length 1. So

$$\sum_k w_t^k e^{\eta(\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t - \ell_t^k)} \;\leq\; e^{\eta^2/8}$$

and hence we obtain a converse to (2):

$$\underbrace{\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t}_{\text{dot loss}} \;\leq\; \underbrace{-\frac{1}{\eta} \ln\left(\sum_k w_t^k e^{-\eta \ell_t^k}\right)}_{\eta\text{-scaled mix loss}} + \frac{\eta}{8}$$

Hedge algorithm

Idea: re-use AA for mix loss, now with *learning rate* $\eta > 0$.

Definition: The *Hedge algorithm* with *learning rate* η plays weights in round t:

$$w_t^k \;=\; \frac{e^{-\eta \sum_{s=1}^{t-1} \ell_s^k}}{\sum_{j=1}^K e^{-\eta \sum_{s=1}^{t-1} \ell_s^j}}. \qquad \text{(Hedge)}$$

or, equivalently, $w_1^k = \frac{1}{K}$ and

$$w_{t+1}^k \;=\; \frac{w_t^k e^{-\eta \ell_t^k}}{\sum_{j=1}^K w_t^j e^{-\eta \ell_t^j}} \qquad \text{(Hedge, incremental)}$$

Hedge analysis

Lemma: The regret of Hedge is bounded by

$$R_T \leq \frac{\ln K}{\eta} + T\frac{\eta}{8}$$

Proof: Applying Hoeffding's Lemma to the loss of each round gives

$$\sum_{t=1}^T \boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t \leq \sum_{t=1}^T \Bigg(\underbrace{\frac{-1}{\eta} \ln \left(\sum_{k=1}^K w_t^k e^{-\eta \ell_t^k} \right)}_{\text{mix loss}} + \underbrace{\eta/8}_{\text{overhead}} \Bigg)$$

The mix loss telescopes, and is bounded by (1) by

$$\sum_{t=1}^T \frac{-1}{\eta} \ln \left(\sum_{k=1}^K w_t^k e^{-\eta \ell_t^k} \right) \leq \min_k \sum_{t=1}^T \ell_t^k + \frac{\ln K}{\eta}. \tag{3}$$

Hedge tuning

Theorem: The Hedge regret bound is minimised at $\eta = \sqrt{\frac{8 \ln K}{T}}$ where it states

$$R_T \leq \sqrt{T/2 \ln K}.$$

Proof: pick η to cancel the derivative.

Note: tuning requires knowledge of the time horizon T.

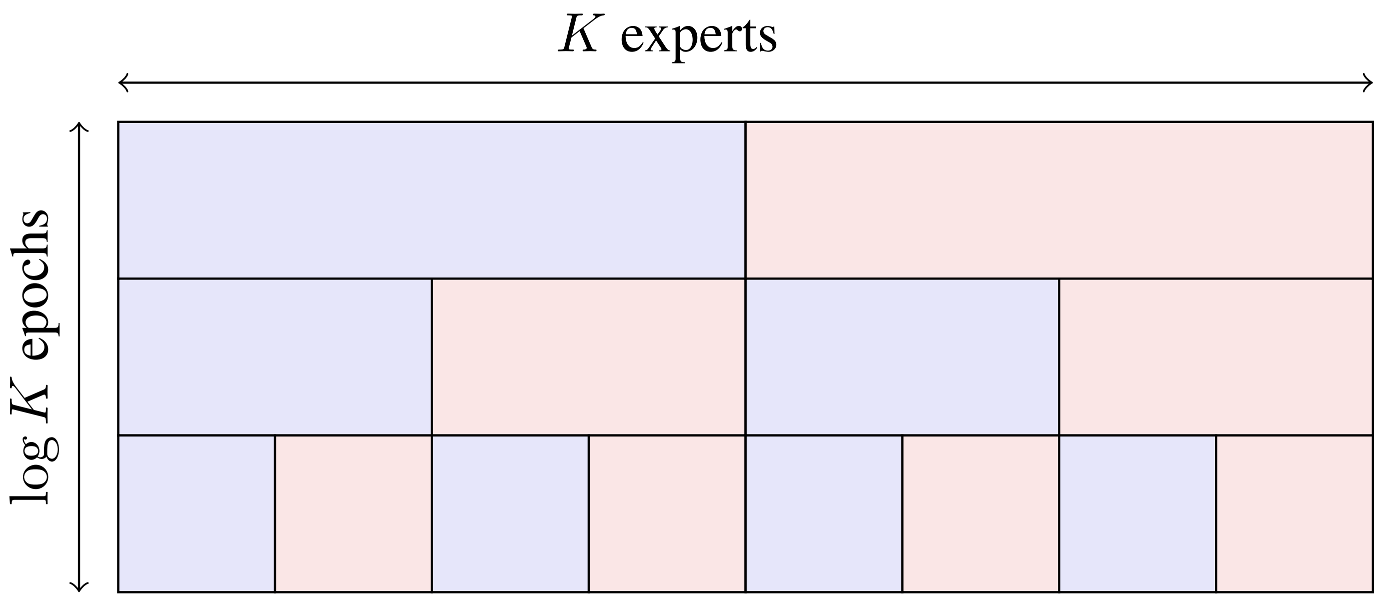
Regret lower bound for the Dot loss game

Is the Hedge algorithm doing actually well?

Theorem: The minimax regret for the dot loss game is $\Omega\left(\sqrt{T \ln K}\right)$.

Proof: We will see in the homework that there is an adversary for the 2-expert game with lower bound $c\sqrt{T}$. Here we boost it to K experts. The construction works by splitting the horizon T into $T/\log K$ epochs. Within each epoch, we will cluster the experts into 2 groups, and apply the 2-expert adversary to each group. This inflicts regret $c\sqrt{T/\log K}$ w.r.t. each expert in the winning group. With K experts, we can split them $\log K$ many times completely independently (see the figure below). The overall regret w.r.t. the expert that is in the wining group in every epoch is

$$R_T \;\geq\; \log(K)c\sqrt{T/\log K} \;=\; c\sqrt{T\log K}$$


K experts
$\log K$ epochs