Machine Learning Theory 2021
Lecture 10

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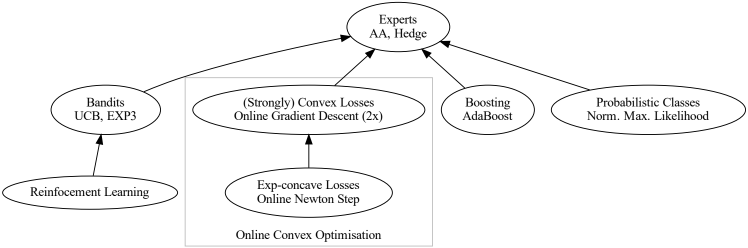
Online Convex Optimisation

- Gradient Descent for Convex Losses
- Online to Batch Conversion
- Gradient Descent for Strongly Convex Losses

homework roulette
in the break

Recap

Overview of Second Half of Course



Material: course notes and selection of sources on MLT website.

Recap: Finite Classes

So far we have seen learning “finite sets”:
Our learning algorithms behave like the **best** among K strategies.

- K-Experts setting
  - Mix loss : Aggregating Algorithm
  - Dot loss : Hedge algorithm
- K-armed bandit settings
  - Adversarial bandit : EXP3
  - Stochastic bandit : UCB

Outlook: Beyond the Finite

What if we want to compete with **infinite** sets?

- Can we?
- How?

In each case, **lower bounds** grow with K: $\ln K$, $\sqrt{T \ln K}$, $\sqrt{TK \ln K}$, $K/\Delta \ln T$. So hopeless in the **unstructured** $K \to \infty$ case.

Today: compete with **continuous** sets of actions, parameterised such that the loss is a **convex** function of the action.

Convexity Review

Convex Functions I : definition

Fix a convex set $\mathcal{U} \subseteq \mathbb{R}^d$.

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Definition

A function $f : \mathcal{U} \to \mathbb{R}$ is convex if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$ and weights $\theta \in [0, 1]$,

$$f(\theta \boldsymbol{x} + (1 - \theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1 - \theta) f(\boldsymbol{y}).$$

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Extends to arbitrary mixtures: $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$ (Jensen).

Convex Functions II : tangent bound

Fact

A differentiable function $f : \mathcal{U} \to \mathbb{R}$ is convex iff for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) \geq \langle \boldsymbol{y} - \boldsymbol{x}, \nabla f(\boldsymbol{x}) \rangle$$

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Symmetrically, $\langle \boldsymbol{y} - \boldsymbol{x}, \nabla f(\boldsymbol{y}) \rangle \geq f(\boldsymbol{y}) - f(\boldsymbol{x})$.

Convex Functions III : sub-gradient

Fact (Sub-gradient)

For any convex $f : \mathcal{U} \to \mathbb{R}$, possibly non-differentiable, and point $\boldsymbol{x} \in \mathcal{U}$, there always exists **some** *vector $\boldsymbol{g} \in \mathbb{R}^d$ such that for all $\boldsymbol{y} \in \mathcal{U}$*

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) \geq \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{g} \rangle$$

Any such vector $\boldsymbol{g}$ is called a **sub-gradient** *(of f at $\boldsymbol{x}$).*

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The gradient of a differentiable function is a sub-gradient.

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The gradient of a differentiable function is a sub-gradient.

We will abuse notation and denote **any** sub-gradient by $\nabla f(\boldsymbol{x})$.

Online Convex Optimisation

Online Convex Optimisation

General yet simple sequential decision problem.

Fix a convex set $\mathcal{U} \subseteq \mathbb{R}^d$.

Protocol

For $t = 1, 2, \ldots$

- Learner chooses a point $\boldsymbol{w}_t \in \mathcal{U}$.
- Adversary reveals convex loss function $f_t : \mathcal{U} \to \mathbb{R}$.
- Learner's loss is $f_t(\boldsymbol{w}_t)$

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Objective:

Regret w.r.t. best point after T rounds:

$$R_T \;=\; \max_{\boldsymbol{u} \in \mathcal{U}} \sum_{t=1}^{T} \left(f_t(\boldsymbol{w}_t) - f_t(\boldsymbol{u}) \right)$$

Example loss functions

Setting	loss function $f_t(\boldsymbol{u})$
Hedge setting	$\boldsymbol{u}^\intercal \boldsymbol{\ell}_t$
Point prediction	$\|\boldsymbol{u} - \boldsymbol{x}_t\|^2$
Regression	$(\boldsymbol{u}^\intercal \boldsymbol{x}_t - y_t)^2$
Logistic regression	$\ln\left(1 + e^{-y_t \boldsymbol{u}^\intercal \boldsymbol{x}_t}\right)$
Hinge loss	$\max\{0, 1 - y_t \boldsymbol{u}^\intercal \boldsymbol{x}_t\}$
Investment	$-\ln(\boldsymbol{u}^\intercal \boldsymbol{x}_t)$
Offline optimisation	$f(\boldsymbol{u})$

Online Gradient Descent (OGD)

Let $\mathcal{U}$ be a closed convex set containing $\mathbf{0}$.

Definition

Online Gradient Descent with learning rate $\eta > 0$ plays

$$\boldsymbol{w}_1 = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{w}_{t+1} = \Pi_{\mathcal{U}}\left(\boldsymbol{w}_t - \eta \nabla f_t(\boldsymbol{w}_t)\right)$$

where $\Pi_{\mathcal{U}}(\boldsymbol{w}) = \arg\min_{\boldsymbol{u} \in \mathcal{U}} \|\boldsymbol{u} - \boldsymbol{w}\|$ is the projection onto $\mathcal{U}$.

Online Gradient Descent (OGD)

Theorem

Let $\|\nabla f_t(\boldsymbol{u})\| \leq G$ *and* $\|\boldsymbol{u}\| \leq D$ *for all* $\boldsymbol{u} \in \mathcal{U}$. *Then*

$$R_T \;=\; \max_{\boldsymbol{u}\in\mathcal{U}} \sum_{t=1}^{T} \left(f_t(\boldsymbol{w}_t) - f_t(\boldsymbol{u})\right) \;\leq\; \frac{1}{2\eta}D^2 + \frac{\eta}{2}TG^2$$

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Corollary

Tuning $\eta = \frac{D}{G\sqrt{T}}$ *results in*

$$R_T^{\boldsymbol{u}} \;\leq\; DG\sqrt{T}$$

Pythagorean Inequality

Lemma (Pythagorean Inequality)

Fix a closed convex set $\mathcal{U} \subseteq \mathbb{R}^d$*. Let* $\boldsymbol{x} \in \mathcal{U}, \boldsymbol{y} \in \mathbb{R}^d$ *and*

$$\hat{\boldsymbol{y}} \;=\; \Pi_{\mathcal{U}}(\boldsymbol{y}) \;=\; \arg\min_{\boldsymbol{u} \in \mathcal{U}} \|\boldsymbol{u} - \boldsymbol{y}\|^2.$$

Then

$$\|\boldsymbol{x} - \hat{\boldsymbol{y}}\|^2 + \|\hat{\boldsymbol{y}} - \boldsymbol{y}\|^2 \;\leq\; \|\boldsymbol{x} - \boldsymbol{y}\|^2$$

NB: not to be confused with **triangle inequality**
$\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|\boldsymbol{x} - \hat{\boldsymbol{y}}\| + \|\hat{\boldsymbol{y}} - \boldsymbol{y}\|$.

Proof of GD regret bound I

We have

$$f_t(\boldsymbol{w}_t) - f_t(\boldsymbol{u}) \;\le\; \langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(w_t) \rangle$$

Moreover,

$$\begin{aligned} \|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2 &= \|\Pi_{\mathcal{U}}\left(\boldsymbol{w}_t - \eta \nabla f_t(\boldsymbol{w}_t)\right) - \boldsymbol{u}\|^2 \\ &\le \|\boldsymbol{w}_t - \eta \nabla f_t(\boldsymbol{w}_t) - \boldsymbol{u}\|^2 \\ &= \|\boldsymbol{w}_t - \boldsymbol{u}\|^2 - 2\eta \langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(\boldsymbol{w}_t) \rangle + \eta^2 \|\nabla f_t(\boldsymbol{w}_t)\|^2 \end{aligned}$$

Hence

$$\langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(\boldsymbol{w}_t) \rangle \;\le\; \frac{\|\boldsymbol{w}_t - \boldsymbol{u}\|^2 - \|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2}{2\eta} + \frac{\eta}{2} \|\nabla f_t(\boldsymbol{w}_t)\|^2$$

Proof of GD regret bound II

Summing over T rounds, we find

$$
\begin{aligned}
R_T^{\boldsymbol{u}} &\leq \sum_{t=1}^{T} \langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(\boldsymbol{w}_t) \rangle \\
&\leq \underbrace{\sum_{t=1}^{T} \frac{\|\boldsymbol{w}_t - \boldsymbol{u}\|^2 - \|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2}{2\eta}}_{\text{telescopes}} + \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(\boldsymbol{w}_t)\|^2 \\
&\leq \frac{\|\boldsymbol{u}\|^2 - \cancel{\|\boldsymbol{w}_{T+1} - \boldsymbol{u}\|^2}}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(\boldsymbol{w}_t)\|^2 \\
&\leq \frac{D^2}{2\eta} + \frac{\eta}{2} T G^2
\end{aligned}
$$

Online to Batch Conversion

Online to Batch Conversion

Goal: obtain an estimator $\hat{\boldsymbol{w}}_T$ with small expected excess risk.

$$\mathop{\mathbb{E}}_{f_1,\ldots,f_T}\left[\mathop{\mathbb{E}}_{f}\left[f(\hat{\boldsymbol{w}}_T) - f(\boldsymbol{u}^*)\right]\right] \;\leq\; \text{small}$$

where the training set $f_1, \ldots, f_T$ and the test sample f are drawn i.i.d. and $\boldsymbol{u}^*$ optimises the risk $\boldsymbol{u} \mapsto \mathbb{E}_f[f(\boldsymbol{u})]$.

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Idea: use online learning algorithm. Given training sample $f_1, \ldots, f_T$, the algorithm picks $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_T$. Let us define the *average iterate estimator*

$$\hat{\boldsymbol{w}}_T \;=\; \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}_t.$$

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Theorem

An online regret bound $R_T \leq B(T)$ implies

$$\mathop{\mathbb{E}}_{\text{iid } f_1,\ldots,f_T, f}\left[f\left(\hat{\boldsymbol{w}}_T\right) - f(\boldsymbol{u}^*)\right] \;\leq\; \frac{B(T)}{T}$$

Online to Batch Proof

$$
\begin{aligned}
&\mathop{\mathbb{E}}_{\text{iid } f_1,\ldots,f_T,f}\left[f\left(\hat{\boldsymbol{w}}_T\right)-f(\boldsymbol{u}^*)\right] \\
&\leq \mathop{\mathbb{E}}_{\text{iid } f_1,\ldots,f_T,f}\left[\frac{1}{T}\sum_{t=1}^{T}\left(f(\boldsymbol{w}_t)-f(\boldsymbol{u}^*)\right)\right] \\
&= \mathop{\mathbb{E}}_{\text{iid } f_1,\ldots,f_T,f}\left[\frac{1}{T}\sum_{t=1}^{T}\left(f_t(\boldsymbol{w}_t)-f_t(\boldsymbol{u}^*)\right)\right] \leq \frac{B(T)}{T}
\end{aligned}
$$

The first step is convexity of f. The last step uses that f and f_t have the same distribution (and $\boldsymbol{w}_t$ is not a function of f_t).

Online Strongly Convex Optimisation

Structure

What if I **know more** about my setting than **convexity of the loss function**? Can I learn faster?

Strongly Convex Case

Definition

A function $f : \mathcal{U} \to \mathbb{R}$ is *strongly convex* to degree $\alpha \geq 0$ if

$$f(\boldsymbol{u}) - f(\boldsymbol{w}) \;\geq\; \langle \boldsymbol{u} - \boldsymbol{w}, \nabla f(\boldsymbol{w}) \rangle + \frac{\alpha}{2} \|\boldsymbol{u} - \boldsymbol{w}\|^2$$

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Example: $f(\boldsymbol{w}) = \|\boldsymbol{w} - \boldsymbol{x}_t\|^2$.

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Example: $f(\boldsymbol{w}) = \|\boldsymbol{w} - \boldsymbol{x}_t\|^2$.

Idea: could this extra knowledge help in the regret rate?

Online Gradient Descent with time-varying learning rate

Definition (OGD with time-varying learning rate)

$$\boldsymbol{w}_1 \;=\; \mathbf{0} \qquad \text{and} \qquad \boldsymbol{w}_{t+1} \;=\; \Pi_{\mathcal{U}}\left(\boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)\right)$$

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$$\boldsymbol{w}_1 = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{w}_{t+1} = \Pi_{\mathcal{U}}\left(\boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)\right)$$

Theorem

For α-strongly convex loss functions, OGD with learning rate $\eta_t = \frac{1}{\alpha t}$ ensures

$$R_T \leq \frac{G^2}{2\alpha}\left(1 + \ln T\right).$$

Proof I

We start with

$$\begin{aligned}
\|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2 &= \|\Pi_{\mathcal{U}}(\boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)) - \boldsymbol{u}\|^2 \\
&\leq \|\boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t) - \boldsymbol{u}\|^2 \\
&= \|\boldsymbol{w}_t - \boldsymbol{u}\|^2 - 2\eta_t \langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(\boldsymbol{w}_t)\rangle + \eta_t^2 \|\nabla f_t(\boldsymbol{w}_t)\|^2
\end{aligned}$$

So that

$$\begin{aligned}
& f_t(\boldsymbol{w}_t) - f_t(\boldsymbol{u}) \\
&\leq \langle \boldsymbol{w}_t - \boldsymbol{u}, \nabla f_t(\boldsymbol{w}_t)\rangle - \frac{\alpha}{2}\|\boldsymbol{w}_t - \boldsymbol{u}\|^2 \\
&\leq \frac{\|\boldsymbol{w}_t - \boldsymbol{u}\|^2 - \|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2 + \eta_t^2 \|\nabla f_t(\boldsymbol{w}_t)\|^2}{2\eta_t} - \frac{\alpha}{2}\|\boldsymbol{w}_t - \boldsymbol{u}\|^2 \\
&= \|\boldsymbol{w}_t - \boldsymbol{u}\|^2 \left(\frac{1}{2\eta_t} - \frac{\alpha}{2}\right) - \frac{\|\boldsymbol{w}_{t+1} - \boldsymbol{u}\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(\boldsymbol{w}_t)\|^2}{2}
\end{aligned}$$

Proof II

Key idea for telescoping:

$$\left(\frac{1}{\eta_{t+1}} - \alpha\right) = \frac{1}{\eta_t}$$

So

$$\eta_{t+1} = \frac{1}{\frac{1}{\eta_t} + \alpha}$$

A good starting point (cancelling the positive term after telescoping) is $\eta_1 = \frac{1}{\alpha}$. This leads to $\eta_t = \frac{1}{\alpha t}$. We then find

$$\begin{aligned}
R_T &= \sum_{t=1}^{T} f_t(\boldsymbol{w}_t) - f_t(\boldsymbol{u}) \\
&\leq \sum_{t=1}^{T} \left(\left\| \boldsymbol{w}_t - \boldsymbol{u} \right\|^2 \left(\frac{1}{2\eta_t} - \frac{\alpha}{2} \right) - \frac{\left\| \boldsymbol{w}_{t+1} - \boldsymbol{u} \right\|^2}{2\eta_t} + \frac{\eta_t \left\| \nabla f_t(\boldsymbol{w}_t) \right\|^2}{2} \right) \\
&\leq \sum_{t=1}^{T} \frac{\left\| \nabla f_t(\boldsymbol{w}_t) \right\|^2}{2\alpha t} \leq \frac{G^2}{2\alpha} (1 + \ln T)
\end{aligned}$$

Conclusion

Tools for learning in convex settings.

- Guaranteed robustness against adversarial losses
- Efficient
- Building block for
  - Learning in non-convex settings (AdaGrad for DNN)
  - Learning in games
  - Non-convex games (GANs)
  - ...