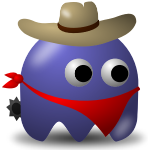
Machine Learning Theory 2024
Lecture 9

Wouter M. Koolen

Download these slides now from elo.mastermath.nl!

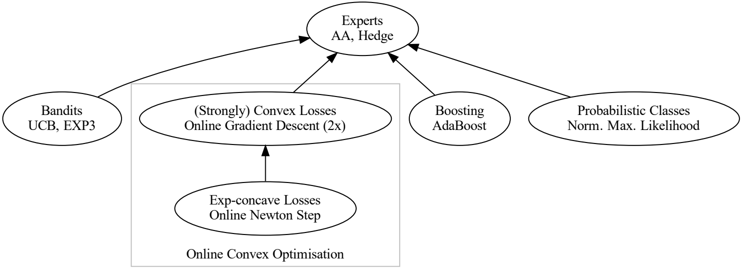
Bandit problems

- Adversarial bandit Setting
- EXP3 Algorithm and analysis
- Stochastic bandit Setting
- UCB Algorithm and analysis



Recap and Bandit Setting

Overview of Second Half of Course



Material: course notes on MLT website.

Recap: Setting

Protocol (Dot Loss Game)

- For $t = 1, 2, \dots$
  - Learner chooses a distribution $\boldsymbol{w}_t \in \triangle_K$ on K "experts".
  - Adversary reveals loss vector $\boldsymbol{\ell}_t \in [0,1]^K$.
  - Learner's loss is the **dot loss** $\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t = \sum_{k=1}^K w_t^k \ell_t^k$

Objective

Regret after T rounds:

$$R_T \;=\; \underbrace{\sum_{t=1}^T \boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t}_{\text{Learner's loss}} - \underbrace{\min_k \sum_{t=1}^T \ell_t^k}_{\text{loss of best expert}}$$

A **good learner** has **small regret**, i.e. it approximately behaves as if it knows the *best expert*.

Recap: Method and Result

Definition (Hedge Algorithm)

The *Hedge algorithm* with *learning rate* η plays weights in round t:

$$w_t^k \;=\; \frac{e^{-\eta \sum_{s=1}^{t-1} \ell_s^k}}{\sum_{j=1}^K e^{-\eta \sum_{s=1}^{t-1} \ell_s^j}}. \qquad \text{(Hedge)}$$

or, equivalently, $w_1^k = \frac{1}{K}$ and

$$w_{t+1}^k \;=\; \frac{w_t^k e^{-\eta \ell_t^k}}{\sum_{j=1}^K w_t^j e^{-\eta \ell_t^j}} \qquad \text{(Hedge, incremental)}$$

Theorem (Hedge Regret Bound)

The regret of Hedge with learning rate $\eta = \sqrt{\frac{8 \ln K}{T}}$ *is at most*

$$R_T \;\leq\; \sqrt{T/2 \ln K}.$$

"Bandits": a (minor?) change of feedback

Learner picks $\boldsymbol{w}_t \in \triangle_K$
Environment determines loss $\boldsymbol{\ell}_t \in [0,1]^K$
Learner sees **full** ℓ_t

↓

Learner picks action $I_t \in [K]$ (possibly at random)
Environment determines loss $\boldsymbol{\ell}_t \in [0,1]^K$
Learner sees **only** $\ell_t^{I_t}$

Radical upgrade: Learner actively controls **which data** are collected.

Applications

- Clinical trials (round=patient, action=treatment)
- Advertising (round=visitor, action=serving specific ad)
- Radio channel selection (wifi)
- ...

Main Questions

- How difficult is it to learn from **partial observations**?
- How should learning algorithms be (re)designed?
  - Obtaining information requires executing sub-optimal actions
  - Exploration/Exploitation trade-off
- What is the effect of the environment model?
  - Adversarial
  - Stochastic

  Different techniques, different complexity (regret rate)

Two Brilliant Ideas

- Importance Weighted Loss Estimates
- Optimism in Face of Uncertainty

Adversarial Bandits

Main Questions

How difficult is it to learn from **partial observations**?

The setup

Protocol (K-armed adversarial bandit)

- Adversary hides $\ell_t^k \in [0,1]$ for all $t \leq T, k \leq K$.
- For $t = 1, 2, \ldots, T$
  - Learner picks arm I_t (typically by sampling $I_t \sim \boldsymbol{w}_t$)
  - Learner observes and incurs *loss* $\ell_t^{I_t}$

Objective:

Expected regret w.r.t. best arm after T rounds:

$$\mathbb{E}[R_T] \; = \; \mathop{\mathbb{E}}_{I_1 \cdots I_T} \left[\sum_{t=1}^{T} \ell_t^{I_t} \right] - \min_k \sum_{t=1}^{T} \ell_t^k$$

Outline of this part

We will prove the following result:

Theorem (Main Adversarial Bandit Result)

There is an algorithm with regret

$$\mathbb{E}[R_T] \leq \cdot\sqrt{2TK \ln K}$$

Ingredients:

- Importance weighted estimates
- Reduction to AA
- Tweaks to AA analysis

Importance Weighted Loss Estimates

- Opponent fixed $\boldsymbol{\ell}_t$.
- We draw $I_t \sim \boldsymbol{w}_t$.
- We see $\ell_t^{I_t}$.

We only see **one entry** of $\boldsymbol{\ell}_t$. Can we still **estimate** the full $\boldsymbol{\ell}_t$?

Definition (Loss Estimate)

The *importance weighted loss estimate* is $\hat{\boldsymbol{\ell}}_t$ with entries $\hat{\ell}_t^k := \frac{\ell_t^{I_t}}{w_t^{I_t}} \mathbf{1}_{I_t = k}$.

Example

Say $K = 4$, $\boldsymbol{w}_t = (0.1, 0.2, 0.3, 0.4)$, $\boldsymbol{\ell}_t = (0.6, 0.7, 0.8, 0.9)$ and sampling from $\boldsymbol{w}_t$ gives $I_t = 3$. Then we see $\ell_t^{I_t} = \ell_t^3 = 0.8$ and form the estimate

$$\hat{\boldsymbol{\ell}}_t = \begin{pmatrix} 0 \\ 0 \\ 0.8/0.3 = 2.66\ldots \\ 0 \end{pmatrix}.$$

Importance Weighted Loss Estimates

Definition (Loss Estimate)

We pick $\hat{\boldsymbol{\ell}}_t$ with $\hat{\ell}_t^k = \frac{\ell_t^{I_t}}{w_t^{I_t}} \mathbf{1}_{I_t = k}$.

Lemma (Unbiased Estimator)

$\mathbb{E}_{I_t \sim \boldsymbol{w}_t}[\hat{\boldsymbol{\ell}}_t] = \boldsymbol{\ell}_t$.

Proof.

For each k

$$\underset{I_t \sim \boldsymbol{w}_t}{\mathbb{E}}[\hat{\ell}_t^k] \;=\; \sum_{I_t=1}^{K} w_t^{I_t} \frac{\ell_t^{I_t}}{w_t^{I_t}} \mathbf{1}_{I_t = k} \;=\; \sum_{I_t=1}^{K} \ell_t^{I_t} \mathbf{1}_{I_t = k} \;=\; \ell_t^k$$

□

Corollary

$\mathbb{E}_{I_t \sim \boldsymbol{w}_t}[\boldsymbol{w}_t^\intercal \hat{\boldsymbol{\ell}}_t] = \boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t = \mathbb{E}_{I_t \sim \boldsymbol{w}_t}[\ell_t^{I_t}]$.

EXP3: AA + scaling + estimation

Slogan: EXP3 is AA applied to η-**scaled importance weighted** losses.

Definition (EXP3 Algorithm)

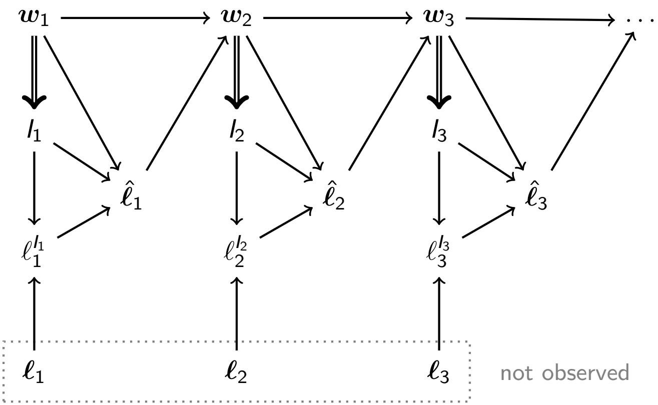
The *EXP3 algorithm* with *learning rate* $\eta > 0$ plays weights in round t:

$$w_t^k = \frac{e^{-\eta \sum_{s=1}^{t-1} \hat{\ell}_s^k}}{\sum_{j=1}^K e^{-\eta \sum_{s=1}^{t-1} \hat{\ell}_s^j}}. \tag{EXP3}$$

or, equivalently, $w_1^k = \frac{1}{K}$ and

$$w_{t+1}^k = \frac{w_t^k e^{-\eta \hat{\ell}_t^k}}{\sum_{j=1}^K w_t^j e^{-\eta \hat{\ell}_t^j}} \tag{EXP3, incremental}$$

Dependency structure



$\longrightarrow$	functional dependency
$\Longrightarrow$	random sampling

EXP3 Analysis: dot/mix loss

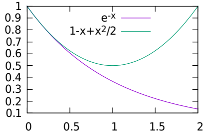
We would like to use the dot/mix loss relationship. For Hedge the losses $\boldsymbol{\ell}_t$ are **bounded**, and we can use Hoeffding's Inequality. But for EXP3 the importance weighted loss estimates $\hat{\boldsymbol{\ell}}_t$ are **not bounded** above. We need another relation.

Lemma

For losses $\hat{\ell}_t^k \geq 0$ and learning rate $\eta > 0$,

$$\underbrace{\sum_{k=1}^K w_t^k \hat{\ell}_t^k}_{\text{dot loss on } \hat{\boldsymbol{\ell}}_t} \leq \underbrace{-\frac{1}{\eta} \ln \left(\sum_{k=1}^K w_t^k e^{-\eta \hat{\ell}_t^k} \right)}_{\text{(scaled) mix loss on } \eta \hat{\boldsymbol{\ell}}_t} + \underbrace{\frac{\eta}{2} \sum_{k=1}^K w_t^k (\hat{\ell}_t^k)^2}_{\text{overhead}} \tag{1}$$

EXP3 Analysis: dot/mix loss



Proof.

For $x \geq 0$, we have $e^{-x} \leq 1 - x + x^2/2$. Hence

$$\begin{aligned} -\ln\left(\sum_{k=1}^K w_t^k e^{-\eta \hat{\ell}_t^k}\right) &\geq -\ln\left(\sum_{k=1}^K w_t^k (1 - \eta \hat{\ell}_t^k + (\eta \hat{\ell}_t^k)^2/2)\right) \\ &= -\ln\left(1 - \eta \sum_{k=1}^K w_t^k \hat{\ell}_t^k + \frac{\eta^2}{2} \sum_{k=1}^K w_t^k (\hat{\ell}_t^k)^2\right) \\ &\geq \eta \sum_{k=1}^K w_t^k \hat{\ell}_t^k - \frac{\eta^2}{2} \sum_{k=1}^K w_t^k (\hat{\ell}_t^k)^2 \end{aligned}$$

Dividing by $\eta > 0$ and moving the rightmost term over gives the lemma. □

EXP3 Analysis: overhead term

Let's study that overhead term in expectation

Lemma

In round t, for the importance weighted loss estimator $\hat{\boldsymbol{\ell}}_t$

$$\mathop{\mathbb{E}}_{I_t \sim \boldsymbol{w}_t}\left[\sum_{k=1}^{K} w_t^k (\hat{\ell}_t^k)^2\right] \;\leq\; K \tag{2}$$

Proof.

By definition of the importance weighted loss estimator

$$\mathop{\mathbb{E}}_{I_t \sim \boldsymbol{w}_t}\left[\sum_{k=1}^{K} w_t^k (\hat{\ell}_t^k)^2\right] \;=\; \sum_{I_t=1}^{K} w_t^{I_t} \sum_{k=1}^{K} w_t^k \left(\frac{\ell_t^{I_t}}{w_t^{I_t}} \mathbf{1}_{I_t=k}\right)^2$$

Only the diagonal $I_t = k$ contributes, and the loss is bounded $\ell_t^{I_t} \in [0,1]$, so

$$= \;\sum_{I_t=1}^{K} w_t^{I_t} w_t^{I_t} \frac{(\ell_t^{I_t})^2}{(w_t^{I_t})^2} \;=\; \sum_{I_t=1}^{K} (\ell_t^{I_t})^2 \;\leq\; K.$$

EXP3 Regret Bound

Theorem

The expected regret of EXP3 with learning rate $\eta > 0$ *is*

$$\mathop{\mathbb{E}}_{I_1 \cdots I_T}\left[\sum_{t=1}^{T} \ell_t^{I_t}\right] - \min_k \sum_{t=1}^{T} \ell_t^k \;\le\; \frac{\ln K}{\eta} + \frac{TK\eta}{2}$$

Corollary

The expected regret of EXP3 with learning rate $\eta = \sqrt{\frac{2 \ln K}{TK}}$ *is*

$$\mathop{\mathbb{E}}_{I_1 \cdots I_T}\left[\sum_{t=1}^{T} \ell_t^{I_t}\right] - \min_k \sum_{t=1}^{T} \ell_t^k \;\le\; \sqrt{2TK \ln K}$$

EXP3 Analysis

Proof.

Sum the mix/dot loss inequality (1) over rounds

$$\sum_{t=1}^{T}\sum_{k=1}^{K} w_t^k \hat{\ell}_t^k \;\leq\; \sum_{t=1}^{T} -\frac{1}{\eta}\ln\left(\sum_{k=1}^{K} w_t^k e^{-\eta \hat{\ell}_t^k}\right) + \sum_{t=1}^{T}\frac{\eta}{2}\sum_{k=1}^{K} w_t^k (\hat{\ell}_t^k)^2$$

Apply the AA regret bound (Lecture 8) to the middle sum

$$\sum_{t=1}^{T}\sum_{k=1}^{K} w_t^k \hat{\ell}_t^k \;\leq\; \min_k \sum_{t=1}^{T} \hat{\ell}_t^k + \frac{\ln K}{\eta} + \sum_{t=1}^{T}\frac{\eta}{2}\sum_{k=1}^{K} w_t^k (\hat{\ell}_t^k)^2$$

Take expectations, and pull the min out using Jensen's Inequality

$$\mathop{\mathbb{E}}_{I_1\cdots I_T}\left[\sum_{t=1}^{T}\sum_{k=1}^{K} w_t^k \hat{\ell}_t^k\right] \;\leq\; \min_k \mathop{\mathbb{E}}_{I_1\cdots I_T}\left[\sum_{t=1}^{T} \hat{\ell}_t^k\right] + \frac{\ln K}{\eta} + \mathop{\mathbb{E}}_{I_1\cdots I_T}\left[\sum_{t=1}^{T}\frac{\eta}{2}\sum_{k=1}^{K} w_t^k (\hat{\ell}_t^k)^2\right]$$

Use unbiasedness and our expected overhead bound (2) to conclude

$$\mathop{\mathbb{E}}_{I_1\cdots I_T}\left[\sum_{t=1}^{T} \ell_t^{I_t}\right] \;\leq\; \min_k \sum_{t=1}^{T} \ell_t^k + \frac{\ln K}{\eta} + T\frac{\eta}{2}K.$$

Conclusion of Adversarial Bandits part

An algorithm that can learn from **partial information** even with **adversarially determined losses**.

Observations:

- Efficient: run time is $O(K)$ per round.
- Regret of EXP3 is $\sqrt{TK \ln K}$ compared to $\sqrt{T \ln K}$ for Hedge in full information setting.
- For $\sqrt{KT}$ lower bound see bonus material.
- Exploration/exploitation. Unsampled arms get 0 estimated loss. So eventually they will get sampled.

Stochastic Bandits

Main Questions

How difficult is it to learn from **partial observations** if we assume the **environment is stochastic**?

Are we back in statistical learning?

Not quite:

- Yes: statistical model for environment
- No (Major): Learner actively controls **which data** are collected
- No (Minor): **sequential** evaluation

A completely different style of algorithm.

Setting

Protocol (K-armed stochastic bandit)

- Environment: distributions $(\nu_1, \dots, \nu_K)$ of arm **rewards**
- For $t = 1, 2, \dots, T$
  - Learner picks arm I_t
  - Learner observes and receives *reward* $X_t \sim \nu_{I_t}$

Definition (Stochastic Bandit Notation)

The **mean reward** of arm k is $\mu^k = \mathbb{E}_{X \sim \nu_k}[X]$. The **best arm** is $i^* = \arg\max_k \mu^k$. The **sub-optimality gap** of arm i is $\Delta_i = \mu^{i^*} - \mu^i$.

Objective

Pseudo-regret after T rounds:

$$\bar{R}_T \;=\; T\mu^{i^*} - \mathop{\mathbb{E}}_{\substack{X_1 \cdots X_T \\ I_1 \cdots I_T}} \left\{ \sum_{t=1}^{T} \mu^{I_t} \right\}$$

Distributional Assumption and Its Consequences

We assume each arm's reward distribution ν_k is **Gaussian** $\mathcal{N}(\mu^k, 1)$.

Lemma (Chernoff Bound)

Let $X_1, \ldots, X_t$ *i.i.d.* $\mathcal{N}(\mu, 1)$ *with average* $\hat{\mu}_t = \frac{1}{t}\sum_{i=1}^{t} X_i$*. For any* $\epsilon \geq 0$

$$\begin{aligned} \mathbb{P}\{\hat{\mu}_t \geq \mu + \epsilon\} \;&\leq\; e^{-t\frac{\epsilon^2}{2}} \quad \textit{and} \\ \mathbb{P}\{\hat{\mu}_t \leq \mu - \epsilon\} \;&\leq\; e^{-t\frac{\epsilon^2}{2}}. \end{aligned} \tag{3a}$$

Distributional Assumption and Its Consequences

We assume each arm's reward distribution ν_k is **Gaussian** $\mathcal{N}(\mu^k, 1)$.

Lemma (Chernoff Bound)

Let $X_1, \ldots, X_t$ i.i.d. $\mathcal{N}(\mu, 1)$ with average $\hat{\mu}_t = \frac{1}{t}\sum_{i=1}^t X_i$. For any $\epsilon \geq 0$

$$\begin{aligned}\mathbb{P}\{\hat{\mu}_t \geq \mu + \epsilon\} &\leq e^{-t\frac{\epsilon^2}{2}} \quad \textit{and} \\ \mathbb{P}\{\hat{\mu}_t \leq \mu - \epsilon\} &\leq e^{-t\frac{\epsilon^2}{2}}.\end{aligned} \tag{3a}$$

Equivalent **confidence interval** statements: for any $\delta \in (0, 1]$,

$$\begin{aligned}\mathbb{P}\left\{\mu \leq \hat{\mu}_t - \sqrt{\frac{2\ln\frac{1}{\delta}}{t}}\right\} &\leq \delta \quad \text{and} \\ \mathbb{P}\left\{\mu \geq \hat{\mu}_t + \sqrt{\frac{2\ln\frac{1}{\delta}}{t}}\right\} &\leq \delta.\end{aligned} \tag{3b}$$

Distributional Assumption and Its Consequences

We assume each arm's reward distribution ν_k is **Gaussian** $\mathcal{N}(\mu^k, 1)$.

Lemma (Chernoff Bound)

Let $X_1, \ldots, X_t$ i.i.d. $\mathcal{N}(\mu, 1)$ with average $\hat{\mu}_t = \frac{1}{t}\sum_{i=1}^{t} X_i$. For any $\epsilon \geq 0$

$$\mathbb{P}\left\{\hat{\mu}_t \geq \mu + \epsilon\right\} \;\leq\; e^{-t\frac{\epsilon^2}{2}} \quad \textit{and}$$
$$\mathbb{P}\left\{\hat{\mu}_t \leq \mu - \epsilon\right\} \;\leq\; e^{-t\frac{\epsilon^2}{2}}. \tag{3a}$$

Equivalent **confidence interval** statements: for any $\delta \in (0, 1]$,

$$\mathbb{P}\left\{\mu \leq \hat{\mu}_t - \sqrt{\frac{2\ln\frac{1}{\delta}}{t}}\right\} \;\leq\; \delta \quad \text{and}$$
$$\mathbb{P}\left\{\mu \geq \hat{\mu}_t + \sqrt{\frac{2\ln\frac{1}{\delta}}{t}}\right\} \;\leq\; \delta. \tag{3b}$$

In fact, we may take **sub-Gaussian** rewards, *defined* to satisfy (3). This includes Gaussian, Bernoulli, non-parametric support $[\pm 1]$, ...

Outline of this part

We will prove the following result:

Theorem (Main Stochastic Bandit Result)

There is an algorithm with pseudo-regret

$$\bar{R}_T \;\leq\; C \left(\sum_{k \neq i^*}^{K} \frac{1}{\Delta_k} \right) \ln T + C'$$

Idea

- For each arm, **estimate** its mean.
- **Empirical estimate** of mean of arm k after t rounds:

$$\hat{\mu}_t^k = \frac{\sum_{s=1}^t X_s 1_{I_s=k}}{N_t^k} \qquad \text{where} \qquad N_t^k = \sum_{s=1}^t 1_{I_s=k}$$

- Uncertainty quantification by means of a confidence interval

$$\mathrm{LCB}_t^k := \hat{\mu}_t^k - \sqrt{\frac{2\alpha \ln(t+1)}{N_t^k}}$$

$$\mathrm{UCB}_t^k := \hat{\mu}_t^k + \sqrt{\frac{2\alpha \ln(t+1)}{N_t^k}}$$

  Claim: True mean $\mu^k \in [\mathrm{LCB}_t^k, \mathrm{UCB}_t^k]$ with near-certainty (probability ≈ 1).
- Strategy: Sample the arm of highest UCB_t

UCB Algorithm

Definition (UCB Algorithm)

In round t, the UCB algorithm with parameter $\alpha > 2$ samples arm

$$I_t \;:=\; \arg\max_k \; \mathrm{UCB}^k_{t-1} \;=\; \arg\max_k \; \hat{\mu}^k_{t-1} + \sqrt{\frac{2\alpha \ln(t)}{N^k_{t-1}}}$$

UCB sets I_t **deterministically** given the past $I_{<t}, X_{<t}$

Optimism in face of uncertainty:
Take the action of highest reward among any plausible bandit model.

Where we are heading

We will show

Theorem (UCB Regret Bound)

UCB with $\alpha > 2$ satisfies

$$\bar{R}_T = \sum_{k=1}^{K} \mathbb{E}[N_T^k]\Delta_k \leq \left(\sum_{k \neq i^*}^{K} \frac{1}{\Delta_i} \right) 8\alpha \ln T + \frac{\alpha}{\alpha - 2} \sum_{k=1}^{K} \Delta_k$$

UCB Analysis

Let $i^* = \arg\max_k \mu^k$ be the index of the arm of highest mean.

If in some round t the algorithm samples a suboptimal arm, $I_t = i \neq i^*$, one of three things must be the case

- We have not sampled arm i often; its confidence width is still large
- Arm i is overestimated (its LCB^i_{t-1} is too high).
- Arm i^* is underestimated (its $\text{UCB}^{i^*}_{t-1}$ is too low).

UCB Analysis

Lemma

If UCB samples suboptimal $I_t = i \neq i^*$ then

(a) $\mathrm{UCB}^{i^*}_{t-1} \leq \mu^{i^*}$ *or*

(b) $\mathrm{LCB}^{i}_{t-1} > \mu_i$ *or*

(c) $N^i_{t-1} < \frac{8\alpha \ln t}{\Delta_i^2}$ *where* $\Delta_i = \mu^{i^*} - \mu^i$.

Proof.

Suppose not. Then

$$\mathrm{UCB}^{i^*}_{t-1} \overset{(a)}{>} \mu^{i^*} = \mu^i + \Delta_i \overset{(c)}{\geq} \mu^i + 2\sqrt{\frac{2\alpha \ln t}{N^i_{t-1}}}$$

$$\overset{(b)}{\geq} \mathrm{LCB}^i_{t-1} + 2\sqrt{\frac{2\alpha \ln t}{N^i_{t-1}}} \overset{\text{def.}}{=} \mathrm{UCB}^i_{t-1}$$

and this contradicts that UCB samples $I_t = i$. □

UCB Analysis

NB: $\mathbb{E}$ scopes the rewards $X_1 \cdots X_T$. UCB picks I_t deterministically.

The pseudo-regret can be rewritten as

$$\bar{R}_T \;=\; T\mu^{i^*} - \mathbb{E}\left[\sum_{t=1}^{T} \mu^{I_t}\right] \;=\; \sum_{k=1}^{K} \mathbb{E}[N_T^k]\Delta_k$$

It hence suffices to bound $\mathbb{E}[N_T^i]$ for suboptimal $i \neq i^*$.

We will show for each $k \neq i^*$

$$\mathbb{E}[N_T^k] \;\leq\; C\frac{8\alpha}{\Delta_k^2} \ln T + C'$$

Slogan: Sub-optimal arms are sampled logarithmically often

UCB Analysis

$$\begin{aligned}
\mathbb{E}[N_T^i] &= \mathbb{E}\left[\sum_{t=1}^{T} 1_{I_t=i}\right] = \sum_{t=1}^{T} \mathbb{P}\{I_t = i\} \\
&= \sum_{t=1}^{T} \mathbb{P}\left\{I_t = i \text{ and } N_{t-1}^i < u\right\} + \sum_{t=1}^{T} \mathbb{P}\left\{I_t = i \text{ and } N_{t-1}^i \geq u\right\} \\
&\leq u + \sum_{t=u+1}^{T} \mathbb{P}\left\{I_t = i \text{ and } N_{t-1}^i \geq u\right\} \\
&\overset{\text{Lemma}}{\leq} u + \sum_{t=u+1}^{T} \mathbb{P}\left\{\mathrm{UCB}_{t-1}^{i^*} \leq \mu^{i^*} \text{ or } \mathrm{LCB}_{t-1}^{i} > \mu_i\right\} \\
&\overset{\text{Union bd.}}{\leq} u + \sum_{t=u+1}^{T} \mathbb{P}\left\{\mathrm{UCB}_{t-1}^{i^*} \leq \mu^{i^*}\right\} + \sum_{t=u+1}^{T} \mathbb{P}\left\{\mathrm{LCB}_{t-1}^{i} > \mu_i\right\}
\end{aligned}$$

where $u = \lceil \frac{8\alpha \ln T}{\Delta_i^2} \rceil \leq \frac{8\alpha \ln T}{\Delta_i^2} + 1.$

Confidence bounds are valid

We need to control two similar deviation events. For the first we will show

$$\sum_{t=u+1}^{T} \mathbb{P}\left\{\mathrm{UCB}_{t-1}^{i^*} \leq \mu^{i^*}\right\} \;\leq\; \sum_{t=u+1}^{T} \frac{1}{t^{\alpha-1}} \;\leq\; \frac{1}{\alpha-2}.$$

Let $\tilde{\mu}_n^i$ be the average of the first n samples from arm i, so that $\hat{\mu}_{t-1}^i = \tilde{\mu}_{N_{t-1}^i}^i$. Then

$$\begin{aligned}
\mathbb{P}\left\{\mathrm{UCB}_{t-1}^{i^*} \leq \mu^{i^*}\right\} &\overset{\text{def}}{=} \mathbb{P}\left\{\hat{\mu}_{t-1}^{i^*} + \sqrt{\frac{2\alpha \ln(t)}{N_{t-1}^{i^*}}} \leq \mu^{i^*}\right\} \\
&= \mathbb{P}\left\{\tilde{\mu}_{N_{t-1}^{i^*}}^{i^*} + \sqrt{\frac{2\alpha \ln(t)}{N_{t-1}^{i^*}}} \leq \mu^{i^*}\right\} \\
&\leq \mathbb{P}\left\{\exists s \in \{1,\dots,t\} : \tilde{\mu}_s^{i^*} + \sqrt{\frac{2\alpha \ln(t)}{s}} \leq \mu^{i^*}\right\} \\
&\overset{\text{union bd}}{\leq} \sum_{s=1}^{t} \mathbb{P}\left\{\tilde{\mu}_s^{i^*} + \sqrt{\frac{2\alpha \ln(t)}{s}} \leq \mu^{i^*}\right\} \\
&= \sum_{s=1}^{t} \mathbb{P}\left\{\mu^{i^*} - \tilde{\mu}_s^{i^*} \geq \sqrt{\frac{2\alpha \ln(t)}{s}}\right\} \overset{\text{Chernoff}}{\leq} \sum_{s=1}^{t} e^{-s(\sqrt{\frac{2\alpha \ln(t)}{s}})^2/2} \;=\; \sum_{s=1}^{t} \frac{1}{t^{\alpha}}
\end{aligned}$$

Overall result

We proved

Theorem (UCB Regret Bound)

UCB with $\alpha > 2$ satisfies

$$\bar{R}_T = \sum_{k=1}^{K} \mathbb{E}[N_T^k]\Delta_k \leq \left(\sum_{k \neq i^*}^{K} \frac{1}{\Delta_i} \right) 8\alpha \ln T + \left(1 + \frac{2}{\alpha - 2} \right) \sum_{k=1}^{K} \Delta_k$$

Conclusion of stochastic bandit part

An algorithm that can learn from **partial information** with **stochastic losses**.

Observations:

- Regret of UCB is $\ln T$ whereas EXP3 is $\sqrt{T}$.
- Regret of UCB is **instance dependent** (through gaps Δ_i)
- Exploration/exploitation mechanism: confidence intervals + optimism
- Matching lower bounds exist (bonus material).