Machine Learning Theory 2025
Lecture 4

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Focus on binary classification:

- Review
- Fundamental theorem: quantitative version
- VC-dimension controls growth function

The Fundamental Theorem of PAC-Learning

Theorem

For binary classification, the following are equivalent:

1. *$\mathcal{H}$ has the* **uniform convergence** *property.*
2. *Any* **ERM** *rule is a successful agnostic PAC-learner for $\mathcal{H}$.*
3. *$\mathcal{H}$ is* **agnostic PAC-learnable**.
4. *$\mathcal{H}$ is* **PAC-learnable**.
5. *Any* **ERM** *rule is a successful PAC-learner for $\mathcal{H}$.*
6. **$\mathcal{H}$ has finite VC-dimension.**

VC-dimension **characterizes** (agnostic) PAC-learnability
and uniform convergence!

- Still to prove: $6 \rightarrow 1$

Uniform Convergence

$\mathcal{H}$ has the **uniform convergence** property:

For finite $m_{\mathcal{H}}^{\mathrm{UC}}(\epsilon, \delta)$,

$$\sup_{h \in \mathcal{H}} |L_{\mathcal{D}}(h) - L_S(h)| \leq \epsilon \qquad \text{with probability} \geq 1 - \delta,$$

whenever $m \geq m_{\mathcal{H}}^{\mathrm{UC}}(\epsilon, \delta)$,

for all $\mathcal{D}, \epsilon, \delta$.

Shattering and VC-Dimension

Definition (Restriction of $\mathcal{H}$ to $\mathcal{C}$)

For finite $\mathcal{C} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathcal{X}$, let $\mathcal{H}_\mathcal{C} = \{\big(h(\boldsymbol{x}_1), \ldots, h(\boldsymbol{x}_k)\big) \mid h \in \mathcal{H}\}$.

- Obtain $\mathcal{H}_\mathcal{C}$ by evaluating hypotheses in $\mathcal{H}$ only on inputs in $\mathcal{C}$.

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Definition (Shattering)

$\mathcal{H}$ **shatters** a finite set $\mathcal{C} \subset \mathcal{X}$ if **$\mathcal{H}$ can classify the elements of $\mathcal{C}$ in all possible ways**, i.e. $|\mathcal{H}_{\mathcal{C}}| = 2^{|\mathcal{C}|}$.

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Definition (Vapnik-Chervonenkis (VC) Dimension)

- VCdim$(\mathcal{H})$ = **maximum size** of finite set $\mathcal{C} \subset \mathcal{X}$ **shattered** by $\mathcal{H}$
- VCdim$(\mathcal{H}) = \infty$ if there is no maximum

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Theorem

Consider binary classification. Suppose $\mathrm{VCdim}(\mathcal{H}) = v < \infty$. *Then there exist absolute constants* $C_1, C_2 > 0$ *such that*

1. *Uniform convergence:*

$$C_1 \frac{v + \ln(1/\delta)}{\epsilon^2} \leq m_{\mathcal{H}}^{UC}(\epsilon, \delta) \leq C_2 \frac{v + \ln(1/\delta)}{\epsilon^2}$$

2. *Agnostic PAC-learning:*

$$C_1 \frac{v + \ln(1/\delta)}{\epsilon^2} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{v + \ln(1/\delta)}{\epsilon^2}$$

3. *PAC-learning:*

$$C_1 \frac{v + \ln(1/\delta)}{\epsilon} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{v \ln(1/\epsilon) + \ln(1/\delta)}{\epsilon}.$$

Uniform Convergence Upper Bound

Upper bound from previous slide that we want to prove:

Theorem

Consider binary classification. Suppose $\mathrm{VCdim}(\mathcal{H}) \leq v < \infty$. *Then there exists an absolute constant* $C > 0$ *such that*

$$\sup_{h \in \mathcal{H}} |L_{\mathcal{D}}(h) - L_S(h)| \leq \epsilon \qquad \text{with probability} \geq 1 - \delta,$$

whenever

$$m \geq C \frac{v + \ln(1/\delta)}{\epsilon^2}.$$

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$$m \geq C \frac{v \, \ln(1/\epsilon) + \ln(1/\delta) + 1}{\epsilon^2}.$$

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- Extra factor $\ln(1/\epsilon)$ is only logarithmic
- It could be avoided with a more involved argument (using a technique called chaining)
- $v = 0 \Rightarrow |\mathcal{H}| = 1$ is trivial, so can assume $v > 0$ w.l.o.g.

Proof Approach

Will define **growth function** $\tau_{\mathcal{H}}(m)$. Then

Part I: Growth function controls uniform convergence:

$$\sup_{h\in\mathcal{H}} |L_{\mathcal{D}}(h) - L_S(h)| \leq c\sqrt{\frac{\ln \tau_{\mathcal{H}}(m)}{m}} + c\sqrt{\frac{\ln(2/\delta)}{m}} \qquad \text{with probability} \geq 1-\delta,$$

Part II: VC-dimension controls growth function:

$$\ln \tau_{\mathcal{H}}(m) \leq v \ln\left(\frac{em}{v}\right) \qquad \text{for } m > v.$$

- Finish: combine Parts I and II, and find lower bound on m s.t. $\sup_{h\in\mathcal{H}} |L_{\mathcal{D}}(h) - L_S(h)| \leq \epsilon$.

Proof Part II:
VC-dimension Controls Growth Function

Growth Function

- **Finite** $\mathcal{H}$ have the uniform convergence property.
- How do we measure the **size of infinite** $\mathcal{H}$?

Growth function: effective size of $\mathcal{H}$ at sample size m:

$$\tau_{\mathcal{H}}(m) = \max_{\mathcal{C} \subset \mathcal{X} : |\mathcal{C}| = m} |\mathcal{H}_{\mathcal{C}}|$$

- Interpretation: How many truly different hypotheses are there when we only observe m inputs $\mathcal{C} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$?
- If $\mathcal{H}$ is finite, then $\tau_{\mathcal{H}}(m) \leq |\mathcal{H}|$

Sauer's Lemma

Growth function: $\tau_{\mathcal{H}}(m) = \max_{|\mathcal{C}|=m} |\mathcal{H}_{\mathcal{C}}|$

Lemma (Sauer-Shelah-Perles)

Suppose $\mathrm{VCdim}(\mathcal{H}) \leq v < \infty$. *Then the growth function is bounded by*

$$\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{v} \binom{m}{i} \leq \begin{cases} 2^m & \text{if } m \leq v \\ \left(\frac{em}{v}\right)^v & \text{if } m > v. \end{cases}$$

- VC-dimension v determines switch from exponential to polynomial growth in m.
- Case $m > v$ is what we need to show for Part II.

Sauer's Lemma

For all $\mathcal{H}$ and all m

$$\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{v} \binom{m}{i},$$

where $\tau_{\mathcal{H}}(m) = \max_{|C|=m} |\mathcal{H}_C|$

Proof:

Will show: For any $\mathcal{H}$ and C of size $|C| = m$

$$|\mathcal{H}_C| \overset{(1)}{\leq} |\{B \subseteq C : \mathcal{H} \text{ shatters } B\}| \overset{(2)}{\leq} \sum_{i=0}^{v} \binom{m}{i}$$

(2): $\mathcal{H}$ shatters $B \Rightarrow |B| \leq v$

nr of sets $B \subseteq C$ with $|B| = i$ is $\binom{m}{i}$

summing over $i = 0, \ldots, v$ implies (2).

(1) $|H_C| \leq |\{B \subseteq C : \mathcal{H} \text{ shatters } B\}|$ for any $|C| = m$ and any $\mathcal{H}$

By induction in m:

<u>$m = 1$:</u>

$|\mathcal{H}_C| = 1 \Rightarrow C$ is not shattered by $\mathcal{H}$
so only $B = \emptyset$ is shattered by $\mathcal{H}$

$\Rightarrow$ r.h.s is 1

$|H_C| = 2 \Rightarrow C$ is shattered and $B = \emptyset$ is shattered

$\Rightarrow$ r.h.s. $= 2$.

$m \geq 2$: Suppose (1) holds for all $m < k$.

To show: (1) holds for $m = k$.

Let $C = \{x_1, \ldots, x_k\}$ be arbitrary.

Want to apply inductive assumption, so define

$$c' = \{x_2, \ldots, x_k\}$$

Let $\mathcal{Y}_0 = \mathcal{H}_{c'} = \{(y_2, \ldots, y_k) \mid \exists y_1 \text{ s.t. } (y_1, y_2, \ldots, y_k) \in \mathcal{H}_c\}$

Then $|\mathcal{Y}_0| \leq |\mathcal{H}_c|$ undercounts $|\mathcal{H}_c|$, because $y_1 = -1$ and $y_1 = +1$ may both satisfy

So let's count how often this happens:

$$\mathcal{Y}_1 = \{(y_2, \ldots, y_k) \mid \forall y_1 \text{ s.t. } (y_1, y_2, \ldots, y_k) \in \mathcal{H}_c\}$$

Thus

$$|\mathcal{H}_c| = |\mathcal{Y}_0| + |\mathcal{Y}_1|$$

Will show:

i) $|\mathcal{Y}_0| \le |\{B \subseteq C : x_1 \notin B,\ \mathcal{H} \text{ shatters } B\}|$

ii) $|\mathcal{Y}_1| \le |\{B \subseteq C : x_1 \in B,\ \mathcal{H} \text{ shatters } B\}|$

So together:

$$|H_C| = |\mathcal{Y}_0| + |\mathcal{Y}_1| \le |\{B \subseteq C : \mathcal{H} \text{ shatters } B\}|,$$

which is to be shown.

<u>i)</u> Recall that

$$C' = \{x_2, \dots, x_k\}, \quad \mathcal{Y}_0 = \mathcal{H}_{C'}$$

$$|\mathcal{Y}_0| = |H_{C'}| \overset{\text{(induction)}}{\le} |\{B \subseteq C' : \mathcal{H} \text{ shatters } B\}|$$

$$= |\{B \subseteq C : x_1 \notin B,\ \mathcal{H} \text{ shatters } B\}|$$

ii) $|\mathcal{Y}_1| \le |\{B \subseteq C : x_1 \in B,\ \mathcal{H} \text{ shatters } B\}|$

Define $\mathcal{H}' = \{h \in \mathcal{H} \mid \exists h' \in \mathcal{H} \text{ s.t.}$ (h and h' agree on C')

$h'(x_i) = h(x_i) \text{ for } i = 2, \dots, k$

$\text{but } h'(x_1) \ne h(x_1)\}$

Observe:

* $\mathcal{H}'$ shatters $B \subseteq C' \iff \mathcal{H}'$ shatters $B \cup \{x_1\}$
* $\mathcal{Y}_1 = \mathcal{H}'_{C'}$ (induction)

$$|\mathcal{Y}_1| = |\mathcal{H}'_{C'}| \le |\{B \subseteq C' : \mathcal{H}' \text{ shatters } B\}|$$
$$= |\{B \subseteq C' : \mathcal{H}' \text{ shatters } B \cup \{x_1\}\}|$$
$$= |\{B \subseteq C : x_1 \in B,\ \mathcal{H}' \text{ shatters } B\}|$$
$$\le |\{B \subseteq C : x_1 \in B,\ \mathcal{H} \text{ shatters } B\}|$$

$\square$

The Final Inequality (Handwritten)

Lemma

$$\sum_{i=0}^{v} \binom{m}{i} \leq \begin{cases} 2^m & \text{if } m \leq v \\ \left(\frac{em}{v}\right)^v & \text{if } m > v \end{cases}$$

Proof: Will use binomial theorem: $(x+y)^m = \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i}$.

$m \leq v$: $\binom{m}{i} = 0$ for $i > m$, so $\sum_{i=0}^{v} \binom{m}{i} = \sum_{i=0}^{m} \binom{m}{i}$. Then apply binomial theorem with $x = y = 1$.

$m > v$: [Simpler proof from Anthony and Bartlett, *Neural Network Learning: Theoretical Foundations*, 1999]

$$\sum_{i=0}^{v} \binom{m}{i} \leq \left(\frac{m}{v}\right)^v \sum_{i=0}^{v} \binom{m}{i} \left(\frac{v}{m}\right)^i \leq \left(\frac{m}{v}\right)^v \sum_{i=0}^{m} \binom{m}{i} \left(\frac{v}{m}\right)^i$$

$$= \left(\frac{m}{v}\right)^v \left(1 + \frac{v}{m}\right)^m \leq \left(\frac{m}{v}\right)^v (e^{v/m})^m = \left(\frac{em}{v}\right)^v$$

(First equality follows from binomial theorem with $x = 1, y = \frac{v}{m}$.)