# Machine Learning Theory 2026
# Lecture 14


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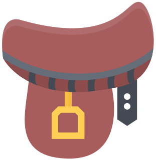
- Two-player zero-sum games
- Nesterov Acceleration from game dynamics
  *Acceleration through Optimistic No-Regret Dynamics*.
  Wang and Abernethy.
  Neural Information Processing Systems (2018).



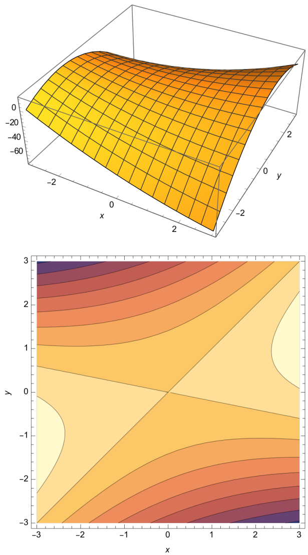
# Two-Player Zero-Sum Games

# Subject

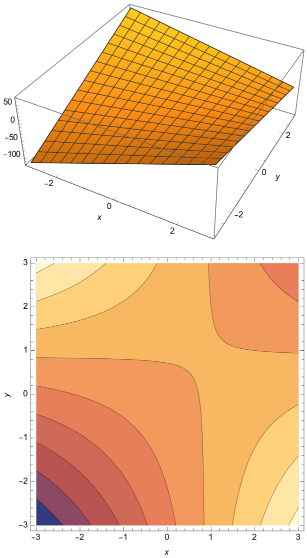
**Two** players. One trying to maximise, one trying to minimise.

What happens when they **both behave optimally**?

# Example two-player objective functions




indefinite quadratic




bi-linear

# Games

Objective function

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convex in $x$, concave in $y$.

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$$V^* - \epsilon \le \inf_x g(x, \bar{y}) \le V^* \le \sup_y g(\bar{x}, y) \le V^* + \epsilon.$$

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Question: how to find $\epsilon$-saddle point?

# Motivation for Saddle Point Computation

- Analysing actual two-player situations
  - Economics
  - Security
  - ...
- Robust learning (Generative Adversarial Networks, ...)
- Applications in offline optimisation
  - Acceleration
  - Constraints (Lagrange multipliers, “primal-dual”, ...)

# Algorithm

Idea: play regret minimisation algorithms for $x$ and $y$.

- Players play $y_t$ and $x_t$.
- Players see loss functions $y \mapsto -g(x_t, y)$ and $x \mapsto +g(x, y_t)$.

Output pair of average iterates: $\left(\frac{1}{T}\sum_{t=1}^{T} x_t, \frac{1}{T}\sum_{t=1}^{T} y_t\right)$.

# Saddle point

Assume the players have regret (bounds) $R_T^x$ and $R_T^y$, i.e.

$$
\begin{aligned}
\sum_{t=1}^{T} +g(x_t, y_t) - \inf_x \sum_{t=1}^{T} +g(x, y_t) \ &\leq \ R_T^x \\
\sum_{t=1}^{T} -g(x_t, y_t) - \inf_y \sum_{t=1}^{T} -g(x_t, y) \ &\leq \ R_T^y
\end{aligned}
$$

**Claim**

$\bar{x}_T = \frac{1}{T} \sum_{t=1}^{T} x_t$ and $\bar{y}_T = \frac{1}{T} \sum_{t=1}^{T} y_t$ form an $\frac{R_T^x + R_T^y}{T}$-saddle point.

# Analysis

$$
\begin{aligned}
V^* &= \inf_x \sup_y g(x,y) \\
&\le \sup_y g(\bar{x}_T, y) \\
&\le \sup_y \frac{1}{T}\sum_{t=1}^{T} g(x_t, y) \\
&\le \frac{1}{T}\sum_{t=1}^{T} g(x_t, y_t) + \frac{R_T^y}{T} \\
&\le \inf_x \frac{1}{T}\sum_{t=1}^{T} g(x, y_t) + \frac{R_T^x + R_T^y}{T} \\
&\le \inf_x g(x, \bar{y}_T) + \frac{R_T^x + R_T^y}{T} \\
&\le \sup_y \inf_x g(x,y) + \frac{R_T^x + R_T^y}{T} \\
&= V^* + \frac{R_T^x + R_T^y}{T}
\end{aligned}
$$

# Nesterov Acceleration

# Offline Optimisation

Starting point: optimisation problem $\inf_x f(x)$.

Regret minimisation algorithm for $\ell_t = f$ gives $O(T^{-1/2})$ suboptimality for average iterate.

Can we do better?

Here we assume that $f$ is $L$-smooth, i.e.

$$\|\nabla f(u) - \nabla f(v)\| \;\leq\; L\|u - v\|$$

(note: converse to strong convexity).

# Fenchel Game

Idea: form **Fenchel game**

$$g(x, y) \;=\; \langle x, y\rangle - f^*(y)$$

where $f^*(y) = \sup_x \langle x, y\rangle - f(x)$ is the **Fenchel conjugate**.

Crux: saddle point for Fenchel game solves minimisation problem :

$$\inf_x \sup_y g(x, y) \;=\; \inf_x \sup_y \langle x, y\rangle - f^*(y) \;=\; \inf_x f^{**}(x) \;=\; \inf_x f(x).$$

# Approximate Saddle point

Moreover, an approximate saddle point gives an approximate minimiser. Recall that

$$V^* \;=\; \inf_x \sup_y g(x, y) \;=\; \inf_x f(x).$$

An $\epsilon$ saddle point $(\bar{x}, \bar{y})$ for the Fenchel game satisfies

$$V^* - \epsilon \;\le\; \inf_x g(x, \bar{y}) \;\le\; V^* \;\le\; \sup_y g(\bar{x}, y) \;\le\; V^* + \epsilon$$

In particular

$$f(\bar{x}) \;=\; \sup_y g(\bar{x}, y) \;\le\; \inf_x f(x) + \epsilon.$$

# Extra Assumption: Smoothness

**Proposition**

*$f$ is smooth $\Leftrightarrow$ $f^*$ is strongly convex.*

We see that the Fenchel game

$$g(x, y) \;=\; \langle x, y \rangle - f^*(y)$$

is *strongly convex* in $y$ and *linear* in $x$.

Idea: exploit strong convexity in Fenchel game.

# Elements of the Approach

The approach combines 4 main ideas

1. Weighting $\alpha_1, \alpha_2, \ldots$ on rounds
2. Order the players: inner player *reacts* to outer player action.
3. Apply Optimistic Follow-The-Leader for $y$ player
4. Apply Online Gradient Descent for $x$ player.

# Weighted rounds

In round $t$ we assign losses scaled by $\alpha_t$

$$x \mapsto \alpha_t g(x, y_t) \qquad \text{and} \qquad y \mapsto -\alpha_t g(x_t, y).$$

We analyse the weighted average iterates

$$\bar{x}_T = \frac{1}{A_T} \sum_{t=1}^{T} \alpha_t x_t \qquad \qquad \bar{y}_T = \frac{1}{A_T} \sum_{t=1}^{T} \alpha_t y_t$$

where $A_t = \sum_{s=1}^{t} \alpha_s$.

# Result for $y$ player

Weighted **Optimistic FTL** plays

$$y_t \;=\; \arg\min_y -\alpha_t g(x_{t-1}, y) + \sum_{s=1}^{t-1} -\alpha_s g(x_s, y)$$

Expanding the Fenchel game, this is

$$y_t \;=\; \nabla f(\tilde{x}_t) \quad \text{where} \quad \tilde{x}_t \;=\; \frac{\alpha_t x_{t-1} + \sum_{s=1}^{t-1} \alpha_s x_s}{A_t}$$

## Theorem

*Optimistic FTL satisfies*

$$\sup_y \sum_{t=1}^{T} \alpha_t \left(g(x_t, y) - g(x_t, y_t)\right) \;\le\; L \sum_{t=1}^{T} \frac{\alpha_t^2}{A_t} \|x_t - x_{t-1}\|^2.$$

# Result for $x$ player

Weighted **Online Gradient Descent** plays

$$x_0 \;=\; 0 \qquad \text{and} \qquad x_t \;=\; x_{t-1} - \gamma \alpha_t \nabla_x g(x, y_t).$$

Expanding the Fenchel Game, this is

$$x_t \;=\; x_{t-1} - \gamma \alpha_t y_t$$

NB: Iterate $x_t$ for round $t$ defined in terms of outer player move $y_t$

**Theorem**

*Let $\|x_*\| \leq D$. Then OGD satisfies*

$$\sum_{t=1}^{T} \alpha_t \left( g(x_t, y_t) - g(x_*, y_t) \right) \;\leq\; \frac{D^2}{\gamma} - \sum_{t=1}^{T} \frac{1}{2\gamma} \left\| x_t - x_{t-1} \right\|^2.$$

The reason we get **negative regret** is that $x$ plays second, with knowledge of $y_t$.

# Combination

In total, we find

$$f(\bar{x}_T) - \min_x f(x) \;\leq\; \frac{1}{A_T}\left(\frac{D^2}{\gamma} + \sum_{t=1}^{T}\left(\frac{\alpha_t^2}{A_t}L - \frac{1}{2\gamma}\right)\|x_t - x_{t-1}\|^2\right).$$

We now tune $\alpha_t, \gamma$ to ensure $\frac{\alpha_t^2}{A_t}L \leq \frac{1}{2\gamma}$. We pick

$$\alpha_t \;=\; t \qquad \text{and} \qquad \gamma \;=\; \frac{1}{4L},$$

for then

$$\frac{\alpha_t^2}{A_t}L \;=\; \frac{t^2}{t(t+1)/2}L \;\leq\; 2L \;=\; \frac{1}{2\gamma}.$$

We conclude

$$f(\bar{x}_T) - \min_x f(x) \;\leq\; \frac{8LD^2}{T^2}.$$

# Final Algorithm: Nesterov Acceleration

Initialise $x_0 = 0$.
For $t = 1, \dots, T$

- $\tilde{x}_t = \frac{\alpha_t x_{t-1} + \sum_{s=1}^{t-1} \alpha_s x_s}{A_t}$
- $y_t = \nabla f(\tilde{x}_t)$
- $x_t = x_{t-1} - \gamma \alpha_t y_t$

Output average iterate

$$\frac{1}{A_T} \sum_{t=1}^{T} \alpha_t x_t$$

# Conclusion of the Lecture

We saw

- Method to learn saddle point in two-player games
- Reduction of offline smooth convex optimisation to saddle point problem

We obtained a hierarchy for offline optimisation

- Convex: $T^{-1/2}$.
- Strongly convex: $T^{-1}$.
- Convex and smooth: $T^{-2}$.

# Conclusion of the Course

We saw

- Stochastic and game-theoretic frameworks for learning
- Ways to characterise the complexity of learning problems
- Algorithms and their analysis

Advanced topics that may interest you

- Reinforcement Learning
- Learning in (strategic) multi-agent problems
- Fairness, Accountability, Transparency
- Beyond convexity (NNs, tensor dec.)

# Conclusion

This conclude the lectures.

- It has been a pleasure
- Good luck for the exam
- If you have an idea that you want to work on ...